Math 739 – Important Groups (homework 4, due Feb 22)

Exercise 4.1. Prove that $\mathrm{rk}\,(F_n) = n$.

Exercise 4.2. Show that a torsion free group that contains an infinite cyclic subgroup of finite index is infinite cyclic. Remark: There is no hint for this problem because I want you to find a short, elegant (and probably new) solution.

Exercise 4.3. Let G be finitely generated. Show that $\mathrm{cd}\, G = 1$ implies that G has at least two ends. Hint: Let Γ be a Cayley graph for G. Relate several cohomology theories of Γ and G with coefficients in $\mathbb{Z}_2$ and $\mathbb{Z}_2 G$.

Exercise 4.4. Let G act on a tree T such that the following conditions are satisfied:

1. G acts freely and transitively on the set of geometric (unoriented) edges.
2. G does not act transitively on the set of vertices.
3. T has no terminal vertices and is not isomorphic to a line.

Let e be an edge in T that connects the vertices v and w. Let G_v and G_w denote the stabilizers of these vertices. Then $G = G_v * G_w$.

Definition 4.5. A <u>tree set</u> is a set T together with a fixpoint free involution $(\overline{-}) : T \to T$ and a binary relation $\to\!\!\!\!\!\succ$ satisfying the following axioms:

1. The relation $\to\!\!\!\!\!\succ$ is a partial ordering.
2. $t \to\!\!\!\!\!\succ t' \Longleftrightarrow \overline{t'} \to\!\!\!\!\!\succ \overline{t}$.
3. For any two elements $t, t' \in T$ exactly one of the following six cases occurs:
$$t = t', \quad \overline{t} = t', \quad t \to\!\!\!\!\!\succ t', \quad \overline{t} \to\!\!\!\!\!\succ t', \quad t \to\!\!\!\!\!\succ \overline{t'}, \quad \overline{t} \to\!\!\!\!\!\succ \overline{t'}.$$
4. For any element $t \in T$, the set $T_t := \{t' \in T \mid t' \to\!\!\!\!\!\succ t\}$ contains no infinite chain $t'_1 \to\!\!\!\!\!\succ t'_2 \to\!\!\!\!\!\succ t'_3 \to\!\!\!\!\!\succ \cdots$.

Exercise 4.6. Show that intervals in tree sets are totally ordered.

Exercise 4.7. Given a tree set T, we construct a graph as follows: the set of oriented edges is T, the vertex set is $\mathcal{V} := T/\sim$ where
$$t \sim t' :\Longleftrightarrow t \to\!\!\!\!\!\succ \overline{t'} \text{ and } \left[t, \overline{t'}\right] = \emptyset,$$
the endpoint map $\tau : T \to \mathcal{V}$ is the canonical projection $T \to T/\sim$, and the initial vertex map $\iota : T \to \mathcal{V}$ is given by $\iota(t) := \tau\left(\overline{t}\right)$. Show that this graph is a tree – here we use the convention that two opposite oriented edges form one geometric edge, thus avoiding bigons.

It suffices to solve, on the average, half of the problems correctly.