Math 749 – Important Groups
Homework 2, due Feb 8

Exercise 2.1. *Let $\Gamma = \Gamma_\Sigma(G)$ be a Cayley graph for the group G with respect to the finite generating system Σ. Show that Γ contains a bi-infinite short-lex geodesic (defined below).*

Any edge path in Γ reads a word in $\Sigma \uplus \Sigma^{-1}$: while you are moving along the path, you pick up the labels of the edges you are going along, when you move with the direction of the edge you read the label, when you are going against the directed edge in Γ you read its inverse.

Fix an order on the set Σ. This induces an ordering on the set of word with letters from Σ: shorter words precede longer words and you use the lexicographic order to break ties. Regarding inverses as lower case variants of the capital letters in Σ, we actually have an order on words in $\Sigma \uplus \Sigma^{-1}$. Every group element is represented by a unique short-lex minimal word. Hence any to vertices in Γ are joined by a unique short-lex minimal edge path. We call those paths short-lex geodesic segments. Note that they are, in fact, geodesic segments.

Now a (bi-infinite) short-lex geodesic is a (bi-infinite) edge path such that every finite sub path is a short-lex geodesic.

Hint: First prove that Γ contains a bi-infinite geodesic.

Exercise 2.2. *The number of ends in a Cayley graph is 0, 1, 2, or ∞: Let $\Gamma := \Gamma_\Sigma(G)$ be the Cayley graph for the group G with with respect to the finite generating set Σ. Show that if Γ has finitely many ends, then the number of ends is ≤ 2. Hint: Assume Γ has three ends. Then there should be a central region where theses ends get tied up. But a Cayley graph looks homogeneous as there is a vertex transitive group action, hence there cannot be a distinguished region.*

Exercise 2.3. *Given the same setup as in (2.2), show that the number of ends (0, 1, 2, or ∞) is independent of the choice of the finite generating system Σ.*

Exercise 2.4. *Let G be finitely generated and H be a subgroup of finite index, which is, therefore, finitely generated, as well. Show that the growth functions for these two groups are weakly equivalent.*

Exercise 2.5. *Show that a finitely generated group with infinitely many ends has exponential growth.*

It suffices to solve, on the average, half of the problems correctly.