6.4 The Dehn-Nielsen Theorem

Recall that any map

$$f : X \to Y$$

induces a homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) \, .$$

This homomorphism depends only on the homotopy class of f. If f is a homotopy equivalence, this map is an isomorphism.

A path p in X induces a "change of basepoint" isomorphism:

$$\begin{aligned} p_* : \pi_1(X, p(1)) &\to \pi_1(X, p(0)) \\ [\gamma] &\mapsto [p \rightarrowtail \gamma \rightarrowtail p^{\mathrm{rev}}] \, . \end{aligned}$$

Fix a basepoint $\underline{\mathtt{P}}$ in the closed oriented surface Σ. For any self-homotopy equivalence $f : \Sigma \to \Sigma$, let p be any path from $\underline{\mathtt{P}}$ to $f(\underline{\mathtt{P}})$. The isomorphism

$$p_* \circ f_* : \pi_1(\Sigma, \underline{\mathtt{P}}) \to \pi_1(\Sigma, \underline{\mathtt{P}})$$

depends on p, but the induced outer automorphism

$$\nu(f) := [p_* \circ f_*] \in \mathrm{Out}(\pi_1(\Sigma, \underline{\mathtt{P}}))$$

does not. Thus, the map $\nu : f \mapsto \nu(f)$ induces a well defined map

$$\nu : M(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma, \underline{\mathtt{P}})) \, .$$

Theorem 6.4.1 (Dehn-Nielsen). *Let Σ be a closed oriented surface with negative Euler characteristic. Then, the map $\nu : M(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma, \underline{P}))$ is an isomorphism of groups. Moreover, every mapping class is realized by a self-homeomorphism of Σ.*

There is a slightly different phrasing of this result in terms of the group of deck transformations. For any homotopy equivalence $f : \Sigma \to \Sigma$ we can choose a lift $\tilde{f} : \tilde{\Sigma} \to \tilde{\Sigma}$. This lift induces an isomorphism $\tilde{f}_* : \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right) \to \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)$ defined by the requirement that

$$\begin{CD}
\tilde{\Sigma} @>{\tilde{f}}>> \tilde{\Sigma} \\
@A{\tau}AA @AA{\tilde{f}_*(\tau)}A \\
\tilde{\Sigma} @>{\tilde{f}}>> \tilde{\Sigma}_2
\end{CD}$$

commutes for every deck transformation $\tau \in \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)$.

Different choices of the lift $\tilde{f}$ yield different isomorphisms, however, these isomorphisms differ only by an inner automorphism. Thus, $\left[\tilde{f}_*\right] \in \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)$ only depends on f and we have a well defined map

$$\begin{aligned} \tilde{\nu} : M(\Sigma) &\to \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big) \\ f &\mapsto \left[\tilde{f}_*\right]. \end{aligned}$$

Exercise 6.4.2. Recall that every choice of a point $\underline{\tilde{P}}$ in the fiber above $\underline{P}$ defines an isomorphism

$$\pi_1(\Sigma, \underline{P}) \to \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right).$$

Show that, independent of the choice of $\underline{\tilde{P}}$, we obtain a well defined isomorphism

$$\Phi : \mathrm{Out}(\pi_1(\Sigma, \underline{P})) \to \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)$$

and that this isomorphism makes the diagram

$$\begin{array}{ccc}
 & & \mathrm{Out}(\pi_1(\Sigma, \underline{P})) \\
 & \overset{\nu}{\nearrow} & \\
M(\Sigma) & & \Big\downarrow \Phi \\
 & \underset{\tilde{\nu}}{\searrow} & \\
 & & \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)
\end{array}$$

commute.

Now (6.4.1) implies:

Corollary 6.4.3. *The map $\tilde{\nu} : M(\Sigma) \to \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)$ is an isomorphism of groups. Moreover, every mapping class is induced by a self-homeomorphism of $\tilde{\Sigma}$.* **q.e.d.**

Corollary 6.4.4. *Let Σ be a closed oriented surface with negative Euler characteristic. Let $\tilde{\Sigma}$ be its universal cover and let $\tilde{\zeta} : \tilde{\Sigma} \to \tilde{\Sigma}$ be a homeomorphism that commutes with all deck transformations, i.e., the following diagram commutes for all deck transformations $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$:*

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{\zeta}} & \tilde{\Sigma} \\ \tau \uparrow & & \uparrow \tau \\ \tilde{\Sigma} & \xrightarrow{\tilde{\zeta}} & \tilde{\Sigma} \end{array}$$

Then $\tilde{\zeta}$ induces a homeomorphism $\zeta : \Sigma \to \Sigma$, which is homotopic to the identity.

Proof. Observe that $\tilde{\zeta}_*$ is the identity. Thus, ζ is in the kernel of $\tilde{\nu}$ and therefore homotopic to the identity. **q.e.d.**

Exercise 6.4.5. Let Σ be a closed oriented surface of Euler characteristic χ. Show that $\pi_1(\Sigma)^{\text{ab}} = \mathbb{Z}^{2-\chi}$.

Corollary 6.4.6. *Let Σ_1 and Σ_2 be two closed oriented surfaces of negative Euler characteristic. Let $\tilde{\Sigma}_i$ be the universal cover of Σ_i, and let $\phi : \mathrm{Cov}\left(\tilde{\Sigma}_1/\Sigma_1\right) \to \mathrm{Cov}\left(\tilde{\Sigma}_1/\Sigma_2\right)$ be an isomorphism. Then there exists a homeomorphism $\tilde{\zeta} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ of the universal covers that makes the diagram*

$$\begin{array}{ccc} \tilde{\Sigma}_1 & \xrightarrow{\tilde{\zeta}} & \tilde{\Sigma}_2 \\ \tau \uparrow & & \uparrow \phi(\tau) \\ \tilde{\Sigma}_1 & \xrightarrow{\tilde{\zeta}} & \tilde{\Sigma}_2 \end{array}$$

commute for each deck transformation $\tau \in \mathrm{Cov}\left(\tilde{\Sigma}_1/\Sigma_1\right)$.

Proof. Note that Σ_1 and Σ_2 are two closed oriented surfaces with isomorphic fundamental groups. By (6.4.5), the two surfaces have the same Euler characteristic and are therefore homeomorphic by the classification of closed oriented surfaces. Thus, there is a homeomorphism

$$\xi : \Sigma_2 \to \Sigma_1,$$

which has a lift

$$\tilde{\xi} : \tilde{\Sigma}_2 \to \tilde{\Sigma}_1.$$

By (6.4.3), the isomorphism $\tilde{\xi}_* \circ \phi : \mathrm{Cov}[\Sigma_1]\tilde{\Sigma}_1 \to \mathrm{Cov}[\Sigma_1]\tilde{\Sigma}_1$ is induced by a homeomorphism $\tilde{\zeta}_1 : \tilde{\Sigma}_1 \to \tilde{\Sigma}_1$. We have the commutative diagram:

$$\begin{array}{ccccc}
\tilde{\Sigma}_1 & \xrightarrow{\tilde{\zeta}_1} & \tilde{\Sigma}_1 & \xleftarrow{\tilde{\xi}} & \tilde{\Sigma}_2 \\
\uparrow{\scriptstyle \tau} & & \uparrow{\scriptstyle \tilde{\xi}_*(\phi(\tau))} & & \uparrow{\scriptstyle \phi(\tau)} \\
\tilde{\Sigma}_1 & \xrightarrow[\tilde{\zeta}_1]{} & \tilde{\Sigma}_1 & \xleftarrow[\tilde{\xi}]{} & \tilde{\Sigma}_2
\end{array}$$

It follows that $\zeta := \xi^{-1} \circ \zeta_1$ satisfies our needs. **q.e.d.**

The remainder of this section is devoted to the proof of the Dehn-Nielsen theorem (6.4.1). We have to check three statements:

1. The map ν is a group homomorphism.

2. This homomorphism is injective.

3. Every outer automorphism of $\pi_1(\Sigma, \underline{P})$ is induced by a homeomorphism of Σ. This statement implies that ν is onto and that every mapping class is realized by a homeomorphism.

The first two statements are proved in the same way as for the torus. However, since we phrased the statement here for self-homotopy equivalences rather than self-homeomorphisms, let us go through the argument again.

First, let us verify that ν is a homomorphism of groups. So let f and h be self-homotopy equivalences of the surface Σ, and let p and q be paths from the basepoint $\underline{P}$ to $f(\underline{P})$ and $h(\underline{P})$, respectively. Then

$$\begin{aligned}
\nu(f)\,\nu(h) &= [p_* \circ f_*]\,[q_* \circ h_*] \\
&= [(p \rightarrowtail f \circ q)_*\,(f \circ h)_*] \\
&= \nu(f \circ h)\,.
\end{aligned}$$

6.4.1 Injectivity

Lemma 6.4.7. *Every closed oriented surface with negative Euler characteristic is aspherical.*

Proof. The universal cover of such a surface is the hyperbolic plane and therefore contractible. **q.e.d.**

Proposition 6.4.8. *Let Σ be a closed oriented surface with negative Euler characteristic, let $f : \Sigma \to \Sigma$ be a homotopy equivalence, and let p be any path from $\underline{P}$ to $f(\underline{P})$. If $p_* \circ f_*$ is an inner automorphism of $\pi_1(\Sigma, \underline{P})$, then f is homotopic to the identity.*

Proof. Let γ be a loop such that the inner automorphism induced by γ equals $p_* \circ f_*$. That is, for any loop γ', we have

$$[\gamma \rightarrowtail \gamma' \rightarrowtail \gamma^{\text{rev}}] = [p \rightarrowtail f \circ \gamma' \rightarrowtail p^{\text{rev}}] .$$

Conjugating by γ^{rev}, we obtain the equation

$$[\gamma'] = [\gamma^{\text{rev}} \rightarrowtail p \rightarrowtail f \circ \gamma' \rightarrowtail p^{\text{rev}} \rightarrowtail \gamma] = [(\gamma^{\text{rev}} \rightarrowtail p) \rightarrowtail f \circ \gamma' \rightarrowtail (\gamma^{\text{rev}} \rightarrowtail p)^{\text{rev}}] . \tag{6.1}$$

Replacing p by $\gamma^{\text{rev}} \rightarrowtail p$, we assume w.l.o.g. that $p_* \circ f_*$ is actually the identity.

We have to construct a homotopy

$$\Phi : \Sigma \times \mathbb{I} \to \Sigma$$

from $f = \Phi(-, 0)$ to the identity $\text{id}_\Sigma = \Phi(-, 1)$. To this end, let D be a genus g standard polygon diagram for Σ, and let $\pi : D \to \Sigma$ be the projection that realizes the identifications by which D describes Σ. We will construct a map

$$\Psi : D \times \mathbb{I} \to \Sigma$$

that will induce the desired homotopy $\Phi : \Sigma \times \mathbb{I} \to \Sigma$.

Note that $\Sigma \times \mathbb{I}$ is obtained from the drum by making identifications along the yellow boundary annulus. These

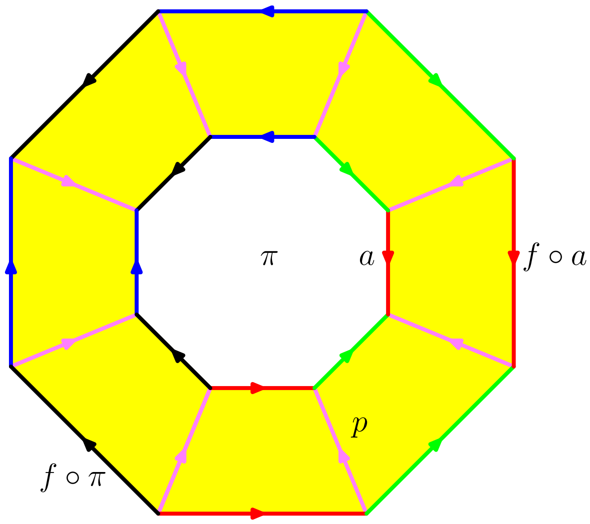

Figure 6.11: The drum (front view).

identifications of faces are induced by the identifications of edges in the polygon diagram D. We will define Ψ in such a way that it is compatible with those identifications. Thus, Ψ will descend to $\Sigma \times \mathbb{I}$.

We define $\Psi : D \times \mathbb{I} \to \Sigma$ as indicated in figure 6.11:

- In the back face $D \times \{1\}$ of the drum $D \times \mathbb{I}$, we define Ψ to be the composition
$$D \times \{0\} \to D \xrightarrow{\pi} \Sigma.$$
- In the front face $D \times \{0\}$, we define Ψ to be the composition
$$D \times \{0\} \to D \xrightarrow{\pi} \Sigma \xrightarrow{f} \Sigma.$$
- The pink edges are of the form $x \times \mathbb{I}$. We map them all to the path p. Formally, on these edges, we define Ψ to be the composition
$$x \times \mathbb{I} \to \mathbb{I} \xrightarrow{p} \Sigma.$$
- Now, we fill in the yellow squares along the boundary of the drum. Note that we already have defined Ψ on the boundary

circles of theses square 2-cells. The boundary of a square whose edge-color is a is mapped to the loop

$$f \circ a \rightarrowtail p \rightarrowtail a^{\text{rev}} \rightarrowtail p^{\text{rev}}.$$

By equation 6.1, this loop is null-homotopic in Σ. Thus, we can extend Ψ to the two yellow squares with this edge-color. Moreover, we can choose the extension so that Ψ is compatible with the face-identifications on the drum.

- As of now, we have defined Ψ on the whole boundary sphere of the drum. By asphericity of Σ, we can extend Ψ to all of $D \times \mathbb{I}$.

Since Ψ is compatible with the face-identifications on $D \times \mathbb{I}$, we can define Φ by the diagram:

$$\begin{array}{ccc} D \times \mathbb{I} & \xrightarrow{\Psi} & \Sigma \\ \downarrow & \nearrow_{\Phi} & \\ \Sigma \times \mathbb{I} & & \end{array}$$

This is the desired homotopy of f and id_Σ. **q.e.d.**

Corollary 6.4.9. *The homomorphism ν is injective.* **q.e.d.**

6.4.2 Surjectivity and the "Moreover,..." Clause

...

Exercise 6.4.10. Prove: In a closed surface with a fixed hyperbolic structure, every closed curve is freely homotopic to a unique closed geodesic – here, a closed geodesic need not be simple.

Definition. Let G be a group with a fixed generating system Σ. The Cayley graph $\Gamma_\Sigma(G)$ is a directed graph whose vertices are the elements of G. For each vertex g and each generator $x \in \Sigma$, there is an edge from g to gx. We ignore the orientation of these edges and define a metric on the vertex set by declaring all edges to have length 1: The metric

$$d_\Sigma : G \times G \to \mathbb{R}$$

is then given by shortest paths – note that $\Gamma(G)$ is connected since Σ generated G.

Exercise 6.4.11. Let G and H be groups generated by the finite generating sets Σ and Ξ, respectively. Let $\varphi : G \to H$ be a group homomorphism. Show that there is a constant C such that for all $g, h \in G$,

$$d_\Xi(\varphi(g), \varphi(h)) \leq C d_\Sigma(g,h).$$

Definition. Two metric space X and Y are called quasi-isometric if there exist two non-negative constants K and C and a function

$$\varphi : X \to Y$$

such that:

1. For all $x, y \in X$,
$$\frac{1}{C} d_X(x,y) - K \leq d_Y(\varphi(x), \varphi(y)) \leq C d_X(x,y) + K.$$
2. Every point in Y is within distance K of the image of φ.

Exercise 6.4.12. Show that quasi-isometry is an equivalence relation on the class of metric spaces.

Exercise 6.4.13. Let Σ be a closed oriented surface with negative Euler characteristic. Show that the Cayley graph of $\pi_1(\Sigma)$ with respect to any finite generating set is quasi-isometric to $\mathbb{H}^2$.

6.5 Calculation of Teichmüller Space

Let Σ be a closed oriented surface of negative Euler characteristic. Teichmüller space $\mathcal{T}_\Sigma$ is the space of all hyperbolic structures on Σ up to equivalence: Let

- $\mathrm{Homeo}(\Sigma)$ be the group of self-homeomorphisms on the torus Σ, and let
- $\mathrm{Homeo}_1(\Sigma)$ be the normal subgroup of those homeomorphisms that are homotopic to the identity. The factor group
- $M(\Sigma) := \mathrm{Homeo}(\Sigma)\,/\mathrm{Homeo}_1(\Sigma)$ is called the mapping class group of Σ.

  Let
- $\mathrm{Isom}(\mathbb{H}^2)$ be the isometry group of the hyperbolic plane. Note that $\mathrm{Isom}(\mathbb{H}^2)$ acts from the left on
- $\mathcal{H}(\Sigma)$, the set of hyperbolic structures on Σ. The action is given by modifying all the charts, appending the isometry $\lambda \in \mathrm{Isom}(\mathbb{H}^2)$.

  Note that $\mathrm{Homeo}(\Sigma)$ acts on $\mathcal{H}(\Sigma)$ from the right as follows: For a homeomorphism $\zeta : \Sigma \to \Sigma$, a given hyperbolic structure $\mathcal{H}$ on Σ and a chart $\varphi : U \to \mathbb{H}^2$ for this structure, define a corresponding chart

  $$\varphi \circ \zeta : \zeta^{-1}(U) \to \mathbb{H}^2.$$

  All these charts form a new atlas for Σ and define a different hyperbolic structure $\mathcal{H}\zeta$. Note that

  $$\zeta : (\Sigma, \mathcal{H}) \to (\Sigma, \mathcal{H}\zeta)$$

  is an equivalence of hyperbolic structures. This action induces an action of $\mathrm{Homeo}(\Sigma)$ on $\mathcal{H}(\Sigma)$.

  The double quotient
- $\mathcal{M}_\Sigma := \mathrm{Isom}(\mathbb{H}^2)\,\backslash\mathcal{H}(\Sigma)\,/\mathrm{Homeo}(\Sigma)$ is called the moduli space of Σ and the quotient
- $\mathcal{T}_\Sigma := \mathrm{Isom}(\mathbb{H}^2)\,\backslash\mathcal{H}(\Sigma)\,/\mathrm{Homeo}_1(\Sigma)$ is called the Teichmüller space of Σ. Note that there is a natural action of $M(\Sigma)$ on $\mathcal{T}_\Sigma$ such that

  $$\mathcal{M}_\Sigma = \mathcal{T}_\Sigma / M(\Sigma)\,.$$

Furthermore, the quotient

- $\mathcal{D}_\Sigma := \mathrm{Isom}(\mathbb{H}^2) \setminus \mathrm{Hom}^{\mathtt{i,d}}(\pi_1(\Sigma, \underline{\mathtt{P}}), \mathrm{Isom}(\mathbb{H}^2))$ is called the <u>deformation space</u> of Σ, where $\mathrm{Hom}^{\mathtt{i,d}}(\pi_1(\Sigma, \underline{\mathtt{P}}), \mathrm{Isom}(\mathbb{H}^2)) :=$ $\{\varphi : \pi_1(\Sigma, \underline{\mathtt{P}}) \to \mathrm{Isom}(\mathbb{H}^2) \mid \varphi$ is injective and has discrete image.$\}$.

Theorem 6.5.1. *The map*

$$\begin{array}{rcl} \Psi : \mathcal{T}_\Sigma & \to & \mathcal{D}_\Sigma \\ [\mathcal{E}] & \mapsto & [\eta^{\delta}_{\mathcal{E}}] \end{array}$$

is a bijection.

Proof of Injectitivity. Suppose we have two hyperbolic structures $\mathcal{H}_1$ and $\mathcal{H}_2$ on Σ such that

$$\left[\eta^{\delta_1}_{\mathcal{H}_1}\right] = \left[\eta^{\delta_2}_{\mathcal{H}_2}\right].$$

Then there is an isometry $\lambda : \mathbb{H}^2 \to \mathbb{H}^2$ such that, for each deck transformation τ, the following diagram commutes:

$$\begin{array}{ccccccc} \tilde{\Sigma} & \xrightarrow{\delta_1} & \mathbb{H}^2 & \xrightarrow{\lambda} & \mathbb{H}^2 & \xleftarrow{\delta_2} & \tilde{\Sigma} \\ \uparrow{\scriptstyle\tau} & & \uparrow{\scriptstyle\eta^{\delta_1}_{\mathcal{H}_1}(\gamma)} & & \uparrow{\scriptstyle\eta^{\delta_1}_{\mathcal{H}_1}(\gamma)} & & \uparrow{\scriptstyle\tau} \\ \tilde{\Sigma} & \xrightarrow[\delta_1]{} & \mathbb{H}^2 & \xrightarrow[\lambda]{} & \mathbb{H}^2 & \xleftarrow[\delta_2]{} & \tilde{\Sigma} \end{array}$$

By (6.4.4), it follows that $\delta_2^{-1} \circ \lambda \circ \delta_1$ induces a homeomorphism $\zeta : \Sigma \to \Sigma$ that is homotopic to the identity. It is easy to check that all these diagrams add up to:

$$\lambda \mathcal{H}_1 \zeta = \mathcal{H}_2$$

Thus, $[\mathcal{H}_1] = [\mathcal{H}_2]$. **q.e.d.**

Exercise 6.5.2. Show that the fundamental group of any non-compact surface is free.

Hint 6.5.3. First, consider the case of a punctured surface Σ, i.e., a closed surface with some discrete set of points removed. Show that there is a graph inside the surface onto which Σ deformation retracts. Then, the fundamental group of the surface is the fundamental group of the graph and hence free.

For the general case, consider a triangulation of the surface Σ. Show that there is a graph Γ inside the 1-skeleton of the triangulation whose complementary components are all infinite, simply-connected, and one-ended: A space X is called one-ended if every compact subset is contained in another compact subset that has a connected complement. Show that Σ deformation retracts onto Γ.

Exercise 6.5.4. Let Σ be a closed, non-orientable surface. Prove that $\pi_1(\Sigma)^{\mathrm{ab}}$ contains an element of order 2.

Corollary 6.5.5. *Two non-homeomorphic closed surfaces have non-isomorphic fundamental groups.*

Proof. This follows from the classification of closed surfaces, (6.4.5), and (6.5.4). Indeed, already the abelianizations of their fundamental groups differ. **q.e.d.**

Proof of surjectivity. Let

$$\eta : \pi_1(\Sigma) = \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right) \to \mathrm{Isom}(\mathbb{H}^2)$$

be an injective homomorphism with discrete image $G := \mathrm{im}(\eta) \leq \mathrm{Isom}(\mathbb{H}^2)$. Note that $G\backslash\mathbb{H}^2$ is a surface with fundamental group G which is isomorphic to $\pi_1(\Sigma)$. Since G cannot be free, $G\backslash\mathbb{H}^2$ is a closed surface. By (6.5.5), the surface $G\backslash\mathbb{H}^2$ is homeomorphic to Σ. Note that $G\backslash\mathbb{H}^2$ comes with a canonical hyperbolic structure. The idea is to pull this one over to Σ.

By (6.4.6), there is a homeomorphism

$$\tilde{\zeta} : \tilde{\Sigma} \to \mathbb{H}^2$$

such that

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{\zeta}} & \mathbb{H}^2 \\ \uparrow \tau & & \uparrow \eta(\tau) \\ \tilde{\Sigma} & \xrightarrow[\tilde{\zeta}]{} & \mathbb{H}^2 \end{array}$$

commutes for each deck transformation τ. Thus, we use $\tilde{\zeta}$ to define a hyperbolic structure on $\tilde{\Sigma}$, which visibly descends to a hyperbolic structure $\mathcal{H}$ on Σ. The homeomorphism $\tilde{\zeta}$ is a developing map for $\mathcal{H}$, and using this developing map, we see that η is the holonomy representation induced by the hyperbolic structure $\mathcal{H}$. **q.e.d.**

6.6 Short Geodesics

We will prove that short simple closed geodesics on a closed hyperbolic surface either coincide or are disjoint. Let us first give a quick an dirty reason why something like that should be true.

Proposition 6.6.1. *Let Σ be a closed hyperbolic surface and let γ_1 and γ_2 be two non-homotopic simple closed geodesics of length $< \operatorname{arcosh}(\frac{5}{4})$. Then, $\gamma_1 \cap \gamma_2 = \emptyset$.*

Proof. Suppose the two loop had an intersection point. We look at the universal cover $\mathbb{H}^2$. We lift the point of intersection and we lift the loops to intersecting geodesic lines, which are the axes of the corresponding deck transformations. The lengths of the loops are precisely the displacements of the two deck transformations.

Let us apply both deck transformations to both geodesics. The key idea is that we cannot get a hyperbolic parallelogram because the angles would not agree – note that we have to find the same angle upstairs at any intersection of axes because this angle can be measured down stairs.

If the displacements are to short, we will have a paralellogram. Contradiction.