Chapter 4

Higher Genus Surfaces

4.1 The Main Result

We will outline two proofs of the main theorem:

Theorem 4.1.1. Let $\Sigma$ be a closed oriented surface of genus $g > 1$. Then every homotopy class of homeomorphisms has a representative $\zeta : \Sigma \to \Sigma$ satisfying one of the following conditions:

elliptic case: The homeomorphism has finite order, i.e., $\zeta^k = \text{id}_\Sigma$.

hyperbolic case: The homeomorphism leaves a pair of geodesic laminations on $\Sigma$ invariant.

parabolic case: There is a non-empty collection of simple closed curves on $\Sigma$ that is left invariant as a subset of $\Sigma$. In this case, a power of $\zeta$ fixes the curves point-wise.

Definition 4.1.2. For a closed oriented surface of genus $g > 1$, the Teichmüller space is defined as

$$T_\Sigma = \left\{ \text{hyperbolic structures on } \Sigma \right\} / \text{Homeo}_1(\Sigma).$$

The main problem to overcome in both proofs is that the action of $M(\Sigma)$ on $T_\Sigma$ is not cocompact. There are two main strategies to overcome this obstacle:
• Restrict your attention to a cocompact subspace of $T_{\Sigma}$.

• Compactify $T_{\Sigma}$ so that the action of $M(\Sigma)$ extends to the compactification.

4.1.1 First Proof: Cutting off Infinity

**Proposition 4.1.3** There is a metric on Teichmüller space $T_{\Sigma}$ such that:

1. $T_{\Sigma}$ is a geodesic metric space.

2. Geodesics are unique.

3. Local geodesics are global.

4. The action of $M(\Sigma)$ on $T_{\Sigma}$ is by isometries.

Thus, $T_{\Sigma}$ is a proper metric space and uniquely geodesic.

**Definition 4.1.4.** Let $X$ be a metric space and $\lambda: X \to X$ be an isometry. The displacement function of $\lambda$ is

\[ D_{\lambda}: X \to \mathbb{R}, \quad x \mapsto d_X(x, \lambda(x)). \]

The displacement of $\lambda$ is

\[ D(\lambda) := \inf_{x \in X} D_{\lambda}(x). \]

The displacement is realized if there is a point $x \in X$ such that

\[ D(\lambda) = D_{\lambda}(x). \]

Fix a homeomorphism

\[ \zeta: \Sigma \to \Sigma, \]

which induces an isometry $\lambda_{\zeta}$ on Teichmüller space by

\[ \lambda_{\zeta}: [\mathcal{H}] \to [\mathcal{H}\zeta]. \]

There are three cases:
• The displacement is realized and equals 0.
• The displacement is realized and strictly positive.
• The displacement is not realized.

The Displacement is Realized and Equals 0
Let $\mathcal{H}$ be a hyperbolic structure on $\Sigma$ such that $[\mathcal{H}] \in \mathcal{T}_{\Sigma}$ realizes the displacement 0. Note that this point is a fixed point of $\zeta$:

$$[\mathcal{H}] = [\mathcal{H}\zeta].$$

Thus there is a homeomorphism $\xi : \Sigma \to \Sigma$ homotopic to the identity such that

$$\mathcal{H}\xi = \mathcal{H}\zeta.$$

Therefore, $\zeta \circ \xi^{-1}$ is an isometry of $(\Sigma, \mathcal{H})$. Since $\xi$ is homotopic to the identity, we conclude that $\zeta$ is homotopic to an isometry of $(\Sigma, \mathcal{H})$. This isometry has finite order:

**Proposition 4.1.5** Any isometry of an oriented closed hyperbolic surface has finite order.

The Displacement is realized and Strictly Positive

Our first goal is to construct a geodesic that is fixed by $\lambda_\zeta$:

**Lemma 4.1.6.** Let $X$ be a geodesic metric space and $\lambda : X \to X$ be an isometry whose displacement is strictly positive and realized at a point $x \in X$. Then

$$l := \bigcup_{k \in \mathbb{Z}} [X, \lambda^k(x)] \lambda^{k+1}(x) = \bigcup_{k \in \mathbb{Z}} \lambda^k [X, x] \lambda(x)$$

is locally a geodesic.

**Proof.** We know that $l$ is geodesic at all points in the interior of $[x, \lambda(x)]$. Since $\lambda$ preserves being locally geodesic, it suffices to show that $l$ is geodesic at $\lambda(x)$.
Consider the midpoint $y$ of $[x, \lambda(x)]$. Observe that

$$D(\lambda) \leq d(y, \lambda(y)) \leq d(y, \lambda(x)) + d(\lambda(x), \lambda(y)) \leq d(x, \lambda(x)) = D(\lambda).$$

Thus $l$ is geodesic at $\lambda(x)$. \hfill q.e.d.

This construction applies to Teichmüller space and yields are global bi-infinite geodesic $C$ by (??.3). Note that this geodesic is invariant with respect to $\lambda_\zeta$.

This is the hyperbolic case:

**Proposition 4.1.7** Every geodesic in Teichmüller space $\mathcal{T}_\Sigma$ gives rise to a pair of transverse geodesic laminations.

**The Displacement is Not Realized**

**Definition 4.1.8.** A metric space is proper if closed balls are compact.

**Exercise 4.1.9.** Show that a metric space is proper if and only if:

$$\text{compact} \iff \text{closed and bounded}$$

**Exercise 4.1.10.** Show that a geodesic metric space is proper if it is complete and locally compact.

**Definition 4.1.11.** A group $G$ acts properly discontinuously on a topological space $X$ if for every compact subset $C \subseteq X$, the set

$$\{g \in G \mid gC \cap C \neq \emptyset\}$$

is finite.

**Remark 4.1.12.** A properly discontinuous action is a topological analogue of an action with finite stabilizers.

We already know that the mapping class group does not act freely on Teichmüller space.
**Promise 4.1.13** Teichmüller space is a complete, locally compact, proper metric space, and the action of the mapping class group acts properly discontinuously on Teichmüller space.

We need a big theorem. For any $\varepsilon > 0$ let $T_\varepsilon$ be the subset of $T_\Sigma$ of those hyperbolic structures for which the length of all closed geodesics in $\Sigma$ are bounded from below by $\varepsilon$. Note that $T_\varepsilon$ is $M(\Sigma)$-invariant.

**Promise 4.1.14 (Mumford’s Compactness Theorem)** For each $\varepsilon > 0$, there is a compact subset $C_\varepsilon \subset T_\Sigma$ such that

$$T_\varepsilon = C_\varepsilon M(\Sigma).$$

In fact, $C_\varepsilon$ can be taken to be a fundamental domain for the action.

Let us choose a sequence of hyperbolic structures $(\mathcal{H}_i)$ such that

$$d([\mathcal{H}_i], [\mathcal{H}_i \zeta]) \to D\lambda_\zeta \quad \text{as } i \to \infty.$$

**Lemma 4.1.15.** There is no $\varepsilon > 0$ such that $[\mathcal{H}_i] \in T_\varepsilon$ for all $i$.

**Proof.** We argue by contradiction. So suppose $[\mathcal{H}_i] \in T_\varepsilon$ for all $i$. Then we can find a sequence $\xi_i \in M(\Sigma)$ such that

$$[\mathcal{H}_i \xi_i] \in C_\varepsilon.$$

Note that the sequence

$$d([\mathcal{H}_i], [\mathcal{H}_i \zeta]) = d([\mathcal{H}_i \xi_i], [\mathcal{H}_i \zeta \xi_i])$$

is bounded. Thus the points

$$[\mathcal{H}_i \zeta \xi_i] = [\mathcal{H}_i \xi_i \circ \xi_i^{-1} \circ \zeta \circ \xi_i]$$

stays within bounded distance from the compact set $C_\varepsilon$. Thus we can pass to a subsequence such that simultaneously

$$[\mathcal{H}_i \xi_i] \to \mathcal{H}_+.$$
and

$$[\mathcal{H}_i \xi_i \circ \xi_i^{-1} \circ \zeta \circ \xi_i] \to \mathcal{H}_+.$$ 

Observe that the isometries $\xi_i^{-1} \circ \zeta \circ \xi_i$ take points close to $\mathcal{H}_+$ to points close to $\mathcal{H}_+$. Since the mapping class group acts properly discontinuously on Teichmüller space, it follows that there are only finitely many elements in $M(\Sigma)$ that do this. By the box principle, one of these occurs infinitely many times in the sequence $\xi_i^{-1} \circ \zeta \circ \xi_i$. Let this isometry be $\xi^{-1} \circ \zeta \circ \xi$. Since

$$d([\mathcal{H}_i],[\mathcal{H}_+]) = D(\zeta)$$

it follows that the displacement of $\zeta$ is realized at

$$[\mathcal{H}_+ \xi^{-1}] .$$

q.e.d.

**Definition 4.1.16.** The *spectrum* of a hyperbolic structure $\mathcal{H}$ on $\Sigma$ is the set

$$\Sigma(\mathcal{H}) := \{ \ln(\gamma) \mid \gamma \text{ is a simple closed geodesic in } \Sigma \} .$$

**Promise 4.1.17** For any hyperbolic surface, closed geodesics of length less than $3 + \sqrt{2}$ do not intersect.

**Promise 4.1.18** Any collection of pairwise non-intersecting non-homotopic loops on a surface of genus $g$ has at most $3g - 3$ elements.

**Corollary 4.1.19.** For any hyperbolic structure $\mathcal{H}$,

$$\left| \Sigma(\mathcal{H}) \cap \left( -\infty, \ln(3 + \sqrt{2}) \right) \right| \leq 3g - 3 .$$

q.e.d.

**Promise 4.1.20** Let $\gamma$ be a simple closed curve on $\Sigma$ that is not homotopically trivial. For each hyperbolic structure $\mathcal{H}$, there is a unique geodesic $\gamma_\mathcal{H}$ homotopic to $\gamma$. Moreover, the map

$$\ell_\gamma : [\mathcal{H}] \mapsto \ln(\text{length of } \gamma_\mathcal{H})$$

is well defined and satisfies the inequality

$$|\ell_\gamma([\mathcal{H}_1]) - \ell_\gamma([\mathcal{H}_2])| \leq d_{\mathcal{T}_\Sigma}([\mathcal{H}_1],[\mathcal{H}_2]) .$$
Choose \( L \) greater than all \( D_{\lambda_i}(|\mathcal{H}_i|) \). Since no \( T_\varepsilon \) contains all \([\mathcal{H}_i]\), it follows that there is an index \( i \) for which
\[
\Sigma(\mathcal{H}_i) = M \uplus N
\]
with
- \( M \neq \emptyset \).
- \( \sup M < \ln(3 + \sqrt{2}) \).
- \( \sup M + L < \inf N \).

We claim that the curves from which the lengths in \( M \) arise form an invariant system. Let \( \Delta \) denote the set of homotopy classes of those closed geodesics.

Observe that
\[
\Sigma(\mathcal{H}) = \Sigma(\mathcal{H}_\zeta) = M \uplus N.
\]
Thus, we may ask whether \( \zeta \) respects the decomposition into \( M \) and \( N \).

The answer is “yes” because of (4.1.20): The curves \( \gamma \) in \( \Delta \) are those with logarithmic length relative to \( \mathcal{H}_i \) in \( M \):
\[
\ell_{\gamma} \mathcal{H}_i \in M.
\]
Since
\[
|\ell_{\gamma} \mathcal{H}_i - \ell_{\gamma} \mathcal{H}_i \zeta| \leq d(\mathcal{H}_i, \mathcal{H}_i \zeta) \leq L,
\]
it follows from \( \sup M + L < \inf N \) that
\[
\ell_{\gamma} \mathcal{H}_i \zeta = \ell_{\zeta \circ \gamma} \mathcal{H} \in M.
\]
Thus, \( \zeta \) permutes the homotopy classes in \( \Delta \). A final fact proves the \( \zeta \) is reducible:

**Promise 4.1.21** If a homeomorphism \( \zeta \) permutes a finite set \( \Delta \) of non-parallel, pairwise disjoint simple closed curves then these homotopy classes can simultaneously realized by simple closed curves which are permuted by a homeomorphism homotopic to \( \zeta \).

### 4.1.2 Second Proof: Compactifying Teichmüller Space