

1. For any r , we have $\bigcup_{i \leq r} J_i^* = \bigcup_{i \leq r} \bigcup_{j=1}^{s_i} P_i^{(j)}$.
2. The interiors of the $P_i^{(j)}$ are pairwise disjoint.
3. Each point is contained in only finitely many $P_i^{(j)}$. Note that by compactness of closed discs, this condition implies that any of these discs meets only finitely many other discs of the sequence.

Thus, we can think of these $P_i^{(j)}$ as a polygonal decomposition of Σ which is easily turned into an honest triangulation.

Hence, we are reduced to proving the existence of the sequence $P_1^{(1)}, P_2^{(1)}, \dots, P_2^{(s_2)}, P_3^{(1)}, \dots, P_3^{(s_3)}, \dots$. Put $P_1^{(1)} := J_1^*$. Suppose already have constructed

$$P_1^{(1)}, P_2^{(1)}, \dots, P_2^{(s_2)}, P_3^{(1)}, \dots, P_3^{(s_3)}, \dots, P_r^{(1)}, \dots, P_r^{(s_r)}.$$

The Jordan domain J_{r+1}^* is chopped up into regions by the boundary curves $\partial(J_i^*)$ for $i \leq r$. Some of these regions might not be discs but contain finitely many holes. We further subdivide and arrive at a decomposition of J_{r+1}^* into finitely many discs. Among these we chose as $P_{r+1}^{(1)}, \dots, P_{r+1}^{(s_{r+1})}$ precisely those that do not contain any interior point of $\bigcup_{i \leq r} J_i^*$.

Of the three requirements our sequence is supposed to meet, only (3) requires proof. So let P be a point in Σ . There is a Jordan domain J_k^* containing P as an interior point. Let U be a neighborhood of P in J_k^* . A disc P_i can intersect U only if $i \leq k$. This establishes (3) and completes the proof. **q.e.d.**

3.4 Geometric Structures

Definition 3.4.1. A differentiable structure on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are differentiable maps.

A complex structure on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are

holomorphic maps. A map $\xi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is holomorphic if it is differentiable and at every point, its derivative is a matrix of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

A Euclidean structure on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are locally Euclidean isometries: A map $\xi: U \rightarrow V$ between open sets in a Euclidean space \mathbb{E} is locally isometric if each point $x \in U$ has an open neighborhood U_x such that

$$\xi|_{U_x} = \lambda|_{U_x}$$

for some isometry $\lambda: \mathbb{E} \rightarrow \mathbb{E}$.

Given a fixed homeomorphic identification of \mathbb{R}^m with hyperbolic m -space, we can define a hyperbolic structure on a manifold as an atlas maximal with respect to the restriction that all coordinate changes are locally hyperbolic isometries. Here locally isometric maps are defined analogously.

Example 3.4.2. Construct the torus by identifying opposite edges of the unit square. This construction imposes a Euclidean structure on the torus. You can realize it with four charts as show in figure 3.5. The blue chart is drawn completely. For each chart, the dashed area in the plane is the range of the chart map. Observe that intersections of chart domains are in general not connected and that the coordinate changes are translations on the components of the intersections.

Example 3.4.3. There is no Euclidean structure on the sphere \mathbb{S}^2 .

Proof. Suppose there was a Euclidean structure. Since the sphere is compact, we could find a cover of the sphere by finitely many flat triangles (i.e., triangles that are completely contained in a

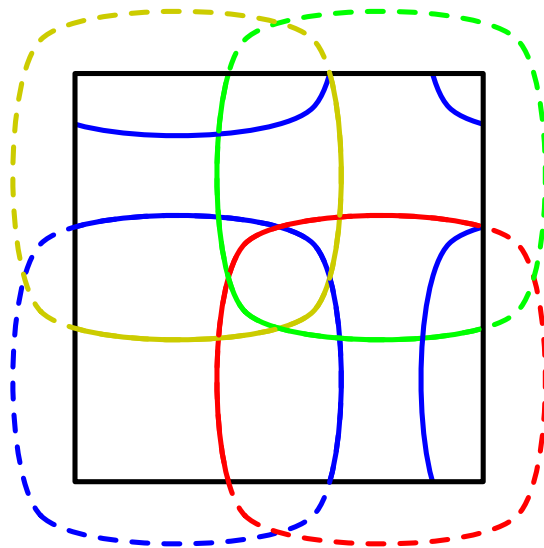


Table 3.5: The “unit square torus”

Euclidean chart and look straight in this chart). We find a common subdivision of all these triangles so that we end up with a flat triangulation of the sphere. Let V , E , and F be the number of vertices, edges, and triangles, respectively.

The angle sum around each vertex is 2π , since this is true in \mathbb{E}^2 . Thus

$$2\pi V = \text{sum of all angles in the triangulation.}$$

On the other hand, we can sum the angles sorted according to the triangles in which they occur. Since the angle sum in a Euclidean triangle is π , we have

$$\pi F = \text{sum of all angles in the triangulation} = 2\pi V.$$

Finally, since each edge has two neighboring triangles and each triangle contains three edges, we have

$$3E = 2F.$$

From this it follows that

$$\begin{aligned}2\pi\chi(\mathbb{S}^2) &= 2\pi V - 2\pi E + 2\pi F \\ &= \pi F - 2\pi\frac{3}{2}F + 2\pi F \\ &= 0\end{aligned}$$

However, $\chi(\mathbb{S}^2) = 2$.

q.e.d.

Remark 3.4.4. This proof contains some insights that are useful:

1. Any closed surface with a geometric structure has a finite triangulation by geodesic triangles.
2. Only a surface with Euler characteristic 0 can support a Euclidean structure.
3. Only a surface with negative Euler characteristic can support a hyperbolic structure.

It follows from the classification of surfaces, that the only closed surface that admits a Euclidean structure is the torus.

3.4.1 $(\mathcal{I}, \mathcal{X})$ -manifolds

Euclidean and hyperbolic structures are just examples of a more general notion.

Definition 3.4.5. Let \mathcal{X} be a fixed m -manifold and let \mathcal{I} be a group of homeomorphisms of \mathcal{X} . A $(\mathcal{I}, \mathcal{X})$ -chart on an m -manifold M is a pair $(U, \varphi: U \rightarrow V \subseteq \mathcal{X})$ where U and V are open sets in M and \mathcal{X} respectively, and $\varphi: U \rightarrow V$ is a homeomorphism. A collection of $(\mathcal{I}, \mathcal{X})$ -charts forms a $(\mathcal{I}, \mathcal{X})$ -atlas if the charts cover M and all coordinate changes

$$\xi: V_0 \rightarrow V_1$$

are locally \mathcal{I} -maps, i.e., for each point $x \in V_0$ there is a homeomorphism $\xi: \mathcal{X} \rightarrow \mathcal{X}$ in \mathcal{I} that equals ξ in an open neighborhood of x .

Exercise 3.4.6. Show that every $(\mathcal{I}, \mathcal{X})$ -atlas for M is contained in a unique maximal $(\mathcal{I}, \mathcal{X})$ -atlas.

Definition 3.4.5 (continued). A $(\mathcal{I}, \mathcal{X})$ -structure for M is a maximal $(\mathcal{I}, \mathcal{X})$ -atlas. A $(\mathcal{I}, \mathcal{X})$ -manifold is a manifold together with a $(\mathcal{I}, \mathcal{X})$ -structure.

Definition 3.4.7. A group \mathcal{I} of homeomorphisms of a topological space \mathcal{X} is rigid for \mathcal{X} if any two homeomorphism ξ_0 and ξ_1 coincide if they coincide on an open subset of \mathcal{X} .

Exercise 3.4.8. Show that the full isometry group of Euclidean m -space is rigid for \mathbb{E}^m .

Definition 3.4.9. A continuous map $f : M_0 \rightarrow M_1$ between Euclidean (hyperbolic) manifolds is geometric if it looks locally like an isometry in local coordinates. That is, for every point $P \in M_0$ there are charts $\varphi_0 : U_0 \rightarrow \mathbb{E}^m$ and $\varphi_1 : U_1 \rightarrow \mathbb{E}^m$ with $P \in U_0$ and $f(P) \in U_1$ such that

$$\varphi_1 \circ f \circ \varphi_0^{-1}$$

is a local isometry in Euclidean space (hyperbolic space).

Two Euclidean (hyperbolic) structures \mathcal{G}_0 and \mathcal{G}_1 on M are equivalent if there is a geometric homeomorphism

$$(M, \mathcal{G}_0) \rightarrow (M, \mathcal{G}_1).$$

Observe that

$$\varphi_1 \circ f \circ \varphi_0^{-1} = \varphi_1 \circ (\varphi_0 \circ f^{-1})^{-1}$$

describes the coordinate change between the given chart in M_1 and a hypothetical chart whose coordinates are given by $\varphi_0 \circ f^{-1}$. Thus, we have a slick way of phrasing this:

Definition 3.4.10. A map $f: M_0 \rightarrow M_1$ is a $(\mathcal{X}, \mathcal{I})$ -map if around each point $P_0 \in M_0$ there exists a chart $(U_0, \varphi_0: U_0 \rightarrow \mathcal{X})$ such that f maps U_0 homeomorphically to an open set $U_1 \subseteq M_1$ that forms a chart together with the coordinate map

$$\varphi_0 \circ f^{-1}: U_1 \rightarrow \mathcal{X}.$$

Remark 3.4.11. It is easy to construct inequivalent Euclidean structures on the torus by rescaling.

Definition 3.4.12. A similarity of a metric space (X, d) is a map

$$\sigma: X \rightarrow X$$

for which there is a constant $L > 0$ such that

$$d(\sigma(x), \sigma(y)) = Ld(x, y).$$

Definition 3.4.13. A continuous map $f: M_0 \rightarrow M_1$ between Euclidean manifolds is a similarity if it looks locally like an similarity in local coordinates. That is, for every point $P \in M_0$ there are charts $\varphi_0: U_0 \rightarrow \mathbb{E}^m$ and $\varphi_1: U_1 \rightarrow \mathbb{E}^m$ with $P \in U_0$ and $f(P) \in U_1$ such that

$$\varphi_1 \circ f \circ \varphi_0^{-1}$$

extends to a similarity of Euclidean space.

Two Euclidean structures \mathcal{G}_0 and \mathcal{G}_1 on M are similar if there is a homeomorphism

$$(M, \mathcal{G}_0) \rightarrow (M, \mathcal{G}_1)$$

that is a similarity.

Exercise 3.4.14. Find two non-similar Euclidean structures on the 2-dimensional torus.

3.4.2 Developing and Holonomy

From now on, we assume that \mathcal{I} is rigid for \mathcal{X} . Let

- P be a point in M and let
- $\varphi : U \rightarrow \mathcal{X}$ be a chart around P . Moreover, let
- $p : \mathbb{I} \rightarrow M$ be a path in M starting at $p(0) = P$.

We will demonstrate how these data give rise to a unique path in \mathcal{X} .

Since \mathbb{I} is compact, there is a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_r = 1$$

and charts

$$(\bigcup_{i=1}^r U_i) \varphi_i : U_i \rightarrow \mathcal{X}$$

such that $\varphi = \varphi_1$ and $p([t_{i-1}, t_i]) \subseteq U_i$. Put

$$x_i := p(t_i)$$

and

$$p_i := p \big|_{[t_{i-1}, t_i]}.$$

Since \mathcal{I} is rigid for \mathcal{X} , there is a unique

$$\xi_i \in \mathcal{I}$$

that extends the coordinate change $\varphi_i \circ \varphi_{i+1}^{-1}$ around a neighborhood of x_i . Thus, we have

$$\varphi_i = \xi_i \varphi_{i+1}$$

around x_i . Then

$$\tilde{p}_i := \xi_1 \circ \cdots \circ \xi_i \circ \varphi_{i+1} \circ p_i : [t_i, t_{i+1}] \rightarrow \mathcal{X}$$

is a path in \mathcal{X} . Note that

$$\begin{aligned}\tilde{p}_{i+1}(t_i) &= \xi_1 \circ \cdots \circ \xi_i \circ \xi_{i+1} \circ \varphi_{i+2}(x_i) \\ &= \xi_1 \circ \cdots \circ \xi_i \circ \varphi_{i+1}(x_i) \\ &= \tilde{p}_i(t_i).\end{aligned}$$

Thus, the path \tilde{p}_{i+1} continues where \tilde{p}_i ends. Therefore, we can connect these pieces and obtain a path

$$\tilde{p}: \mathbb{I} \rightarrow \mathcal{X}.$$

Definition 3.4.15. This path \tilde{p} is called the continuation of p along the chart $\varphi: U \rightarrow \mathcal{X}$.

Lemma 3.4.16. *The continuation \tilde{p} only depends on p and $\varphi: U \rightarrow \mathcal{X}$.*

It does not depend on the partition or the chain of charts covering the partition.

Proof. First, we prove that the continuation does not depend on the chain of charts once a partition is fixed. So suppose $\psi: V \rightarrow \mathcal{X}$ is another sequence of chart with $\varphi = \psi_1$ and $p([t_{i-1}, t_i]) \subseteq V_i$. Let ζ_i be the induced sequence in \mathcal{I} . We claim that

$$\xi_1 \circ \cdots \circ \xi_i \circ \varphi_{i+1} = \zeta_1 \circ \cdots \circ \zeta_i \circ \psi_{i+1}. \quad (3.1)$$

The proof is by induction. The statement is true for $i=0$ since $\varphi = \varphi_1 = \psi_1$.

Now suppose (3.1) holds for i . Then the diagram (3.6) commutes in the component of $p(t_i)$ in $U_i \cap V_i \cap U_{i+1} \cap V_{i+1}$. Now, the path p_{i+1} is connected whence the diagram commutes all along the segment up to $p(t_{i+1})$.

Now suppose, we also change the partition. Since any two partitions have a common refinement and we can use the charts for a given partition for any refinement, as well, we can find a common third to see that the continuations arising this way are actually equal. **q.e.d.**

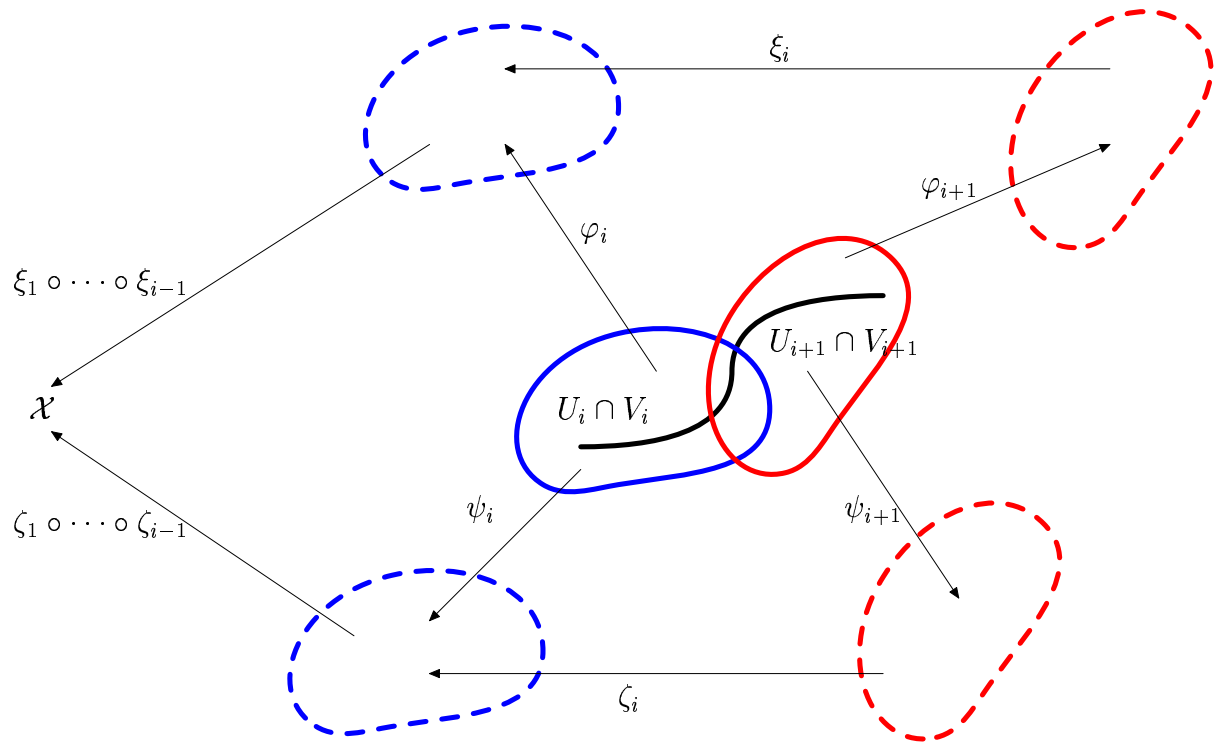
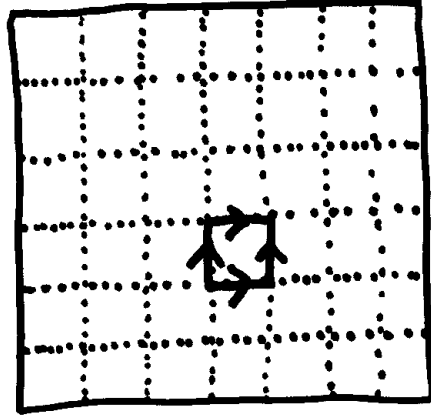


Table 3.6: The diagram commutes near the path because the path is connected.

Proposition 3.4.17. *Let $\varphi: U \rightarrow \mathcal{X}$ be a chart and let $p: \mathbb{I} \rightarrow M$ and $q: \mathbb{I} \rightarrow M$ be two paths whose endpoints coincide. If p and q are homotopic relative to their endpoints then the endpoints of their continuations coincide and their continuations are homotopic relative to their endpoints.*

Proof. This is obvious if the homotopy stays within one coordinate chart. If this is not the case, subdivide the homotopy into small pieces.



q.e.d.

Corollary 3.4.18. *Let M be a 1-connected $(\mathcal{I}, \mathcal{X})$ -manifold and $\varphi : U \rightarrow \mathcal{X}$ be a chart. Then there is a unique $(\mathcal{I}, \mathcal{X})$ -map*

$$\tilde{\varphi} : M \rightarrow \mathcal{X}$$

that extends the chart φ .

Proof. Let P_0 be a fixed point in U . For any point $P \in M$, there is a path p connecting P_0 to P . Continuing φ along p , we obtain a value $\tilde{\varphi}(P)$ that is actually independent of p since any two path from P_0 to P are homotopic relative to their endpoints. The map

$$\tilde{\varphi} : M \rightarrow \mathcal{X}$$

is clearly a $(\mathcal{I}, \mathcal{X})$ -map. This proves existence.

To see that $\tilde{\varphi}$ is unique, suppose that $\nu : M \rightarrow \mathcal{X}$ is a $(\mathcal{I}, \mathcal{X})$ -map that extends φ . Note that for each open set V in M , the restriction

$$\nu|_V : V \rightarrow \mathcal{X}$$

is a chart since ν is a $(\mathcal{I}, \mathcal{X})$ -map. Thus, we may use these charts to compute the continuation of φ along p . Since we just restrict from a globally defined map, there is no need ever to move path segments to make them fit. Thus all the patching homeomorphisms taken from \mathcal{I}

will be trivial and the continuation we obtain from these charts will evaluate to $\nu(P)$ at that end of p . However, the continuation is independent of the charts used to compute it. Thus

$$\nu(P) = \tilde{\varphi}(P). \quad \mathbf{q.e.d.}$$

Observation 3.4.19. *Let M be a connected $(\mathcal{I}, \mathcal{X})$ -manifold, and let*

$$\nu_0, \nu_1 : M \rightarrow \mathcal{X}$$

be two $(\mathcal{I}, \mathcal{X})$ -maps. Then there is a unique element $\xi \in \mathcal{I}$ such that

$$\nu_1 = \xi \circ \nu_0.$$

Proof. Let us prove uniqueness first: Suppose we had two elements ξ and ζ such that

$$\nu_1 = \xi \circ \nu_0 \quad \text{and} \quad \nu_1 = \zeta \circ \nu_0.$$

Then ξ and ζ agree at least on a small open set in the image of ν_0 . Thus, by rigidity of \mathcal{I} , we have $\xi = \zeta$.

For existence, observe that ν_0 and ν_1 are both charts (with fairly big domains). Therefore, $\nu_1 \nu_0^{-1}$ is a coordinate change. Hence it is locally represented by elements of \mathcal{I} . Since M is connected, there is an element ξ that represents the coordinate change globally. **q.e.d.**

Observation 3.4.20. *Let $f : N \rightarrow M$ be a local homeomorphism and M be a $(\mathcal{I}, \mathcal{X})$ -manifold. Then there is a unique $(\mathcal{I}, \mathcal{X})$ -structure on N that renders f to be a $(\mathcal{I}, \mathcal{X})$ -map. This structure is induced by charts of the form*

$$V \xrightarrow{f} U \xrightarrow{\varphi} \mathcal{X}$$

where V is open and homeomorphic via f to the domain of the chart $\varphi : U \rightarrow \mathcal{X}$.

In particular, if $\pi : \bar{M} \rightarrow M$ is a covering space of a $(\mathcal{I}, \mathcal{X})$ -manifold M , then \bar{M} is a $(\mathcal{I}, \mathcal{X})$ -manifold in a canonical way.