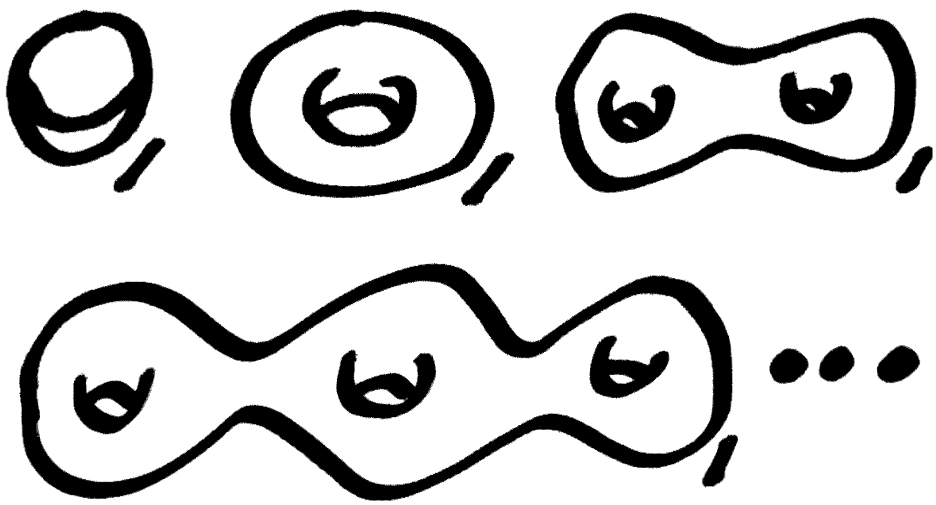
1 Preview

This lecture will deal predominantly with closed orientable surfaces, i.e., one of the following:



So let Σ be one of these. We will:

1. Classify all curves in Σ

   - up to homotopy.
   - up to isotopy (for simple curves).[1]

   We shall see that, for surfaces, there is no difference in these two notions. We are lead to the <u>fundamental group</u> $\pi_*(\Sigma)$.

2. Classify all homeomorphisms $\varphi : \Sigma \to \Sigma$

   - up to homotopy.
   - up to isotopy.[2]

   This leads us to the <u>mapping class group</u> $\mathcal{M}(\Sigma)$.

3. We will show that $\mathcal{M}(\Sigma) = \mathrm{Out}(\pi_*(\Sigma))$.

[1]An <u>isotopy</u> of an embedding is a homotopy that stays an embedding during the deformation.

[2]An <u>isotopy</u> of a homeomorphism is a homotopy that stays a homeomorphism.

To state our other goals, let us briefly discuss the torus.

Fact 1.1. *Let* $\mathbb{T} = \mathbb{T}^2$ *be the two-dimensional torus.*

1. $\pi_*(\mathbb{T}) = C_\infty \times C_\infty$.

2. $\mathcal{M}(\mathbb{T}) = \mathrm{GL}_2(\mathbb{Z})$.

Elements $M \in \mathrm{GL}_2(\mathbb{Z})$ *can be sorted according to trace. There are three cases*

$|\mathrm{tr}(M)| < 2$ *(elliptic): In this case,* M *has finite order. The corresponding homeomorphisms are called finite.*

$|\mathrm{tr}(M)| = 2$ *(parabolic): In this case, there is a homeomorphism* $\varphi : \mathbb{T} \to \mathbb{T}$ *represented by* M *that leaves a simple closed curve on* $\mathbb{T}$ *fixed. The homeomorphisms in this isotopy class are called reducible*

To see that there is a fixed curve, we consider the universal cover $\mathbb{R} \times \mathbb{R}$*. Here the matrix defines a linear map* $M : \mathbb{R}^2 \to \mathbb{R}^2$ *which preserves the lattice* $\mathbb{Z}^2$*. Therefore the map directly descends to the map on* $\mathbb{T}$*. Since the trace has absolute value* 2*, the matrix* M *has a double eigenvalue at* 1 *or* -1*. The corresponding eigenspace has rational slope and therefore descends to a closed curve on* $\mathbb{T}$*, which is fixed.*

$|\mathrm{tr}(M)| > 2$ *(hyperbolic): In this case, the matrix* M *represents a so called Anosov-homeomorphism. The matrix* M *has two different eigenvalues. The corresponding eigenspaces have irrational slope. Thus they descend to a pair of space filling curves on* $\mathbb{T}$ *that intersect transversally. This is the prime example of a pair of transverse foliations. The homeomorphism is stretching in the direction of one foliation and shrinking with respect to the other one.*

Our first main goal will be to extend this to other surfaces:

The Nielsen-Thurston Classification. *Let Σ be a higher genus surface. Then every homeomorphism is either finite, i.e., isotopic to a homeomorphism of finite order, or reducible, i.e., isotopic to a homeomorphism that leaves a multi-curve fixed, or Pseudo-Anosov to be defined later. Roughly speaking, a Pseudo-Anosov homeomorphism locally looks like a map*

$$(x, y) \mapsto (Cx, \frac{y}{C}).$$

To achieve this goal, we will need to introduce the Teichmüller space of Σ. The mapping class group acts on Teichmüller space, and it is a good understanding of this action that will allow us to prove the Nielsen-Thurston classification.

Another space upon which the mapping class group acts is the Curve Complex. This is a simplicial complex whose vertices are simple closed curves on Σ. A set of those forms a simplex, if they do not intersect pairwise.

We will study these and other spaces to prove various result about the mapping class group. In particular, we will see that it is finitely generated.

Along the way, we will have to prove many classical results of planar geometric topology:

1. The Jordan Curve Theorem: A simple closed curve separates the plane into two regions one of which is bounded.

2. Schönflies' Theorem: The bounded region in the Jordan Curve Theorem actually is a disc.

3. The Hauptvermutung for surfaces: Given a finite system of arcs in a surface, there always is a homeomorphism that takes them simultaneously to polygonal arcs – here polygonal is meant to be detected in the universal cover: either $\mathbb{R}^2$ or the hyperbolic plane.