Exercise 11.1. Prove: A polygon diagram describes an orientable surface if and only if, for each edge-color \( a \), the two edges of color \( a \) are oriented oppositely in the boundary circle of the polygon diagram.

Exercise 11.2. Show that any non-orientable surface has a one-vertex-diagram whose boundary reads the colors

\[
\bullet \rightarrow a_1 \bullet \rightarrow a_1 \bullet \rightarrow a_2 \bullet \rightarrow a_2 \bullet \rightarrow a_3 \bullet \rightarrow a_3 \bullet \rightarrow \ldots \bullet \rightarrow a_g \rightarrow \bullet \rightarrow a_g \rightarrow
\]

for some \( g \geq 0 \).

Exercise 11.3. Prove: In a closed surface with a fixed hyperbolic structure, every closed curve is freely homotopic to a unique closed geodesic – here, a closed geodesic need not be simple.

Definition. Let \( G \) be a group with a fixed generating system \( \Sigma \). The Cayley graph \( \Gamma_{\Sigma}(G) \) is a directed graph whose vertices are the elements of \( G \). For each vertex \( g \) and each generator \( x \in \Sigma \), there is an edge from \( g \) to \( gx \). We ignore the orientation of these edges and define a metric on the vertex set by declaring all edges to have length 1: The metric

\[
d_{\Sigma} : G \times G \to \mathbb{R}
\]

is then given by shortest paths – note that \( \Gamma(G) \) is connected since \( \Sigma \) generated \( G \).

Exercise 11.4. Let \( G \) and \( H \) be groups generated by the finite generating sets \( \Sigma \) and \( \Xi \), respectively. Let \( \varphi : G \to H \) be a group homomorphism. Show that there is a constant \( C \) such that for all \( g, h \in G \),

\[
d_{\Xi}(\varphi(g), \varphi(h)) \leq Cd_{\Sigma}(g, h).
\]

Definition. Two metric space \( X \) and \( Y \) are called quasi-isometric if there exist two non-negative constants \( K \) and \( C \) and a function

\[
\varphi : X \to Y
\]

such that:

1. For all \( x, y \in X \),

\[
\frac{1}{C}d_X(x, y) - K \leq d_Y(\varphi(x), \varphi(y)) \leq Cd_X(x, y) + K.
\]

2. Every point in \( Y \) is within distance \( K \) of the image of \( \varphi \).

Exercise 11.5. Show that quasi-isometry is an equivalence relation on the class of metric spaces.

Exercise 11.6. Let \( \Sigma \) be a closed oriented surface with negative Euler characteristic. Show that the Cayley graph of \( \pi_1(\Sigma) \) with respect to any finite generating set is quasi-isometric to \( \mathbb{H}^2 \).

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.