Definition. Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is a homotopy equivalence if there is a map $h : Y \to X$ such that $h \circ f : X \to X$ is homotopic to the identity on $X$ and $f \circ h : Y \to Y$ is homotopic to the identity on $Y$. (I.e., a homotopy equivalence is a map that induces an isomorphism in the homotopy category :-)

Two spaces $X$ and $Y$ are called homotopy equivalent if there is a homotopy equivalence $f : X \to Y$.

Note that any homotopy equivalences $f : X \to Y$ induces an isomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x)).$$

In particular, any self homotopy equivalence $f : X \to X$ induces an outer automorphism $\nu(f)$ of $\pi_1(X)$ in the same way that a self homeomorphism does.

Exercise 8.1. Let $f : T \to T$ be a self homotopy equivalence of the torus. Assume $\nu(f)$ is the class of inner automorphisms of $\pi_1(T)$. (This is, given a path $p$ from $t$ to $f(t)$, the homomorphism $p_* \circ f_*$ is an inner automorphism of $\pi_1(T, t)$.) Prove that $f$ is homotopic to the identity.

Exercise 8.2. Let $f : T \to T$ be a self homotopy equivalence of the torus. Show that $f$ is homotopic to a homeomorphism.

Exercise 8.3. Construct an example of a self homotopy equivalence of a finite graph $\Gamma$ that is not homotopic to a homeomorphism.

Exercise 8.4. Let $\Gamma$ be a finite graph. Let $f_0, f_1 : \Gamma \to \Gamma$ be two self homotopy equivalences of $\Gamma$. Prove that $f_0$ is homotopic to $f_1$ if $\nu(f_0) = \nu(f_1)$.

Exercise 8.5. Let $R_n$ be the graph with one vertex $v$ and $n$ loops attached to the vertex. Show that every automorphism of $F_n := \pi_1(R_n, v)$ arises as an $f_*$ for some homotopy equivalence

$$f : R_n \to R_n.$$

Exercise 8.6. Let $T_1$ and $T_2$ be two tori, and let $\phi : \text{Cov} \left( \tilde{T}_1 / T_1 \right) \to \text{Cov} \left( \tilde{T}_1 / T_2 \right)$ be an isomorphism. Show that there exists a homeomorphism $\tilde{\zeta} : \tilde{T}_1 \to \tilde{T}_2$ of the universal covers such that $\tilde{\zeta} \circ \tau = \phi(\tau) \circ \tilde{\zeta}$ holds for each deck transformation $\tau \in \text{Cov} \left( T_1 \right)$.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.