Math 661 – Geometric Topology (homework 7, due Oct 25)

**Exercise 7.1 (Correction).** Let \( \mathcal{P}_x(X) \) denote the set
\[
\mathcal{P}_x(X) := \{ p : \mathbb{I} \to X \mid p \text{ is continuous and } p(0) = x \}
\]
and let \( \sim \) be the equivalence relation of paths being homotopic relative to endpoints. Let the topology on \( \mathcal{P}_x(X)/\sim \) be defined by basic open sets
\[
U_{p,V} := \{ q \mid q \text{ is a path in } V \text{ starting at } p(1) \}
\]
where \( p : \mathbb{I} \to X \) is a path and \( V \) is an open neighborhood of the endpoint \( p_1 \).

Let
\[
p : \mathbb{I} \to \mathcal{P}_x(X)/\sim
\]
be a path starting at the base point of \( \mathcal{P}_x(X)/\sim \) which is class of the constant path \( p_0 : s \mapsto x \). Prove that any representative of the class \( p_1 \) is homotopic relative endpoints to the path
\[
q : \mathbb{I} \to X
\]
\[
t \mapsto p_t(1).
\]

**Exercise 7.2.** Show that the isometry groups of Euclidean spaces are linear. That is, show that for every \( m \), the group \( \text{Isom}(\mathbb{E}^m) \) is isomorphic to a subgroup of \( \text{GL}_n(\mathbb{R}) \) for some \( n \).

**Exercise 7.3.** Let \( X \) and \( Y \) be topological spaces. Assume \( X \) is discrete. Then, every function from \( X \) to \( Y \) is continuous. Thus in the realm of sets, we have the identity
\[
C(X,Y) = \text{Map}(X,Y).
\]
However, in the realm of topological spaces, the left hand carries the compact-open topology whereas the right hand comes with the product topology—recall that \( \text{Map}(X,Y) \) is just a product of “\( X \)-many” copies of \( Y \). Show that the identity above holds in the realm of topological spaces, i.e., show that the compact open topology agrees with the product topology.

**Exercise 7.4.** Consider the action of \( \text{GL}_2(\mathbb{R}) \) on the real projective line \( \mathbb{RP}_1 \). Let \( M \in \text{GL}_2(\mathbb{R}) \) be a matrix with \( \text{tr}(M) > 2 \). Prove that there are precisely two fixed points \( x_- \) and \( x_+ \) in \( \mathbb{RP}_1 \). Moreover show that the dynamics of \( M \) is as follows: For any pair of open neighborhoods \( U_i \) of \( x_i \), there is a power \( n \) such that
\[
M^n(x) \in U_+ \text{ for any point } x \notin U_-.
\]
Thus, \( x_+ \) is attracting and \( x_- \) is repelling.

Each problem is worth 5 points, but you can earn at most 15 points with this assignment.

Late homework will not be accepted.