Math 661 – Geometric Topology (homework 6, due Oct 18)

Exercise 6.1. Let Σ be a Euclidean surface. Show that there is a cover of Σ by charts $U_i \xrightarrow{\varphi_i} \mathbb{E}^2$ such that for each i

$$d(P, Q) = d(\varphi_i(P), \varphi_i(Q)).$$

On the left hand, the distance is the distance of shortest paths in Σ connecting P and Q, whereas on the right hand, the distance is the standard Euclidean metric.

This exercise says that there is an atlas whose charts are isometrically true pictures of their domains. Note that a big chart on the torus will usually not give the right distances for points near the boundary since a shortest path might leave the chart.

Exercise 6.2. Prove that an isometry of a Euclidean space $\mathbb{E}$ has a fixed point if it is of finite order. (Hint: It might be easier to show the stronger statement that any finite group of isometries of a Euclidean space $\mathbb{E}$ has a global fixed point.)

Exercise 6.3. Let M be a compact Euclidean Manifold. Show that the fundamental group $\pi_1(M)$ is torsion free, i.e., it does not contain any elements of finite order.

Exercise 6.4. Let $\mathcal{P}_{\underline{\mathrm{x}}}(X)$ denote the set

$$\mathcal{P}_{\underline{\mathrm{x}}}(X) := \{p : \mathbb{I} \to X \mid p \text{ is continuous and } p(0) = \underline{\mathrm{x}}\}$$

and let $\sim$ be the equivalence relation of paths being homotopic relative to endpoints. Let the topology on $\mathcal{P}_{\underline{\mathrm{x}}}(X) / \sim$ be defined by basic open sets

$$U_{p,V} := \{pq \mid q \text{ is a path in } V \text{ starting at } p(1)\}$$

where $p : \mathbb{I} \to X$ is a path and V is an open neighborhood of the endpoint p_1.

Let

$$\begin{aligned} p : \mathbb{I} &\to \mathcal{P}_{\underline{\mathrm{x}}}(X) / \sim \\ t &\mapsto p_t \end{aligned}$$

be a path. Prove that any representative of the class p_1 is homotopic relative endpoints to the path

$$\begin{aligned} q : \mathbb{I} &\to X \\ t &\mapsto p_t(1). \end{aligned}$$

Exercise 6.5. Show that any isometry of the plane $\mathbb{E}^2$ is either a rotation or a translation or a glide reflection, i.e., a reflection followed by a possibly trivial translation along the axis of the reflection.

Each problem is worth 5 points, but you can earn at most 15 points with this assignment.

Late homework will not be accepted.