Math 661 – Geometric Topology (homework 5, due Oct 11)

Exercise 5.1 (Hauptvermutung for Surfaces). Show that any two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ of the surface Σ are combinatorially equivalent. You may assume that Σ is compact. (I.e., I will give full credit for a solution to this case.)

Hint: I am thinking of two possible attacks.

1. Show that there is a triangulation $\mathcal{T}'$ of Σ that is combinatorially equivalent to both $\mathcal{T}_0$ and $\mathcal{T}_1$ by virtue of homework (4.6). To find $\mathcal{T}'$, carefully analyse Rado's proof of triangulability.
2. Show that there is a homeomorphism $\varphi : \Sigma \to \Sigma$ such that $\varphi(\mathcal{T}_0)$ and $\mathcal{T}_1$ are combinatorially equivalent by homework (4.6). To construct φ, embark on the early exercises of homework 4.

Warning: I did not carefully check whether these ideas actually work out.

Exercise 5.2. Show that for two $(\mathcal{I}, \mathcal{X})$-maps $\nu_0, \nu_1 : \bar{M} \to \mathcal{X}$, there is a unique $g \in \mathcal{I}$ such that

$$\eta_{\nu_0}(h) = g\eta_{\nu_1}(h)\, g^{-1}.$$

Thus, two holonomies differ by an inner automorphism of $\mathcal{I}$.

Exercise 5.3. Give an example of a local homeomorphism that is not a covering projection.

Exercise 5.4. Let $\bar{X}$ be a path connected topological space and G be a group that acts topologically free on $\bar{X}$. Then

$$\pi : \bar{X} \to X := {}_{G}\backslash^{\bar{X}}$$

turns $\bar{X}$ into a covering space for the quotient space X.

Exercise 5.5. Let Y be a topological space. Suppose the diagram

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\bar{f}_0} & \bar{X} \\ \downarrow & & \downarrow \pi \\ Y \times [0,1] & \xrightarrow[f]{} & X \end{array}$$

commutes. We saw that the exists a unique function

$$\bar{f} : Y \times [0,1] \to \bar{X}$$

such that

$$\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{\bar{f}_0} & \bar{X} \\
\downarrow & \overset{\bar{f}}{\nearrow} & \downarrow \pi \\
Y \times [0,1] & \xrightarrow[f]{} & X
\end{array}$$

commutes. Show that $\bar{f}$ is continuous.

Exercise 5.6 (Covering Homotopy Lemma). Let $\bar{X}$ be a covering space for X with covering projection $\pi : \bar{X} \to X$. Let x_0 and x_1 be two points in X connected by two paths $p, q : \mathbb{I} \to X$ starting at x_0 and ending at x_1 that are homotopic relative to their endpoints. Furthermore, let $\bar{x}_0$ be a point in the fiber over x_0. Consider the two lifts $\bar{p}$ and $\bar{q}$ starting at $\bar{x}_0$. Prove that $\bar{p}(1) = \bar{q}(1)$ and that the paths $\bar{p}$ and $\bar{q}$ are homotopic relative to their endpoints.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.