A SUPERPOSITION IN POLYTROPIC FLOW

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ABSTRACT Simple polytropic waves which propagate in three space dimensions may be superposed provided that the angle between the directions of propagation has a precise value which depends on the ratio of specific heats. The velocities add, and densities combine nonlinearly. Numerical evidence is given and the solutions are proved mathematically.

Consider the polytropic Euler equations in three space dimensions

$$u_t + u \cdot \nabla u + \rho^{-1} \nabla p = 0,$$

$$\rho_t + \mathrm{div}(\rho u) = 0, \quad p = k\rho^\gamma$$

We assume $1 < \gamma < 3$, k is constant, and set $a = \frac{\gamma - 1}{2}$.
Let $f(s,t)$ be any smooth solution to the inviscid Burgers equation

$$f_t + (1+a) f f_s = 0$$

and v any constant unit vector. There are known solutions expressible as

$$\begin{aligned} u(x,t) &= f(x \cdot v, t) v \\ \rho(x,t) &= \left(\frac{a}{\sqrt{k\gamma}} f(x \cdot v, t) \right)^{\frac{1}{a}}. \end{aligned}$$

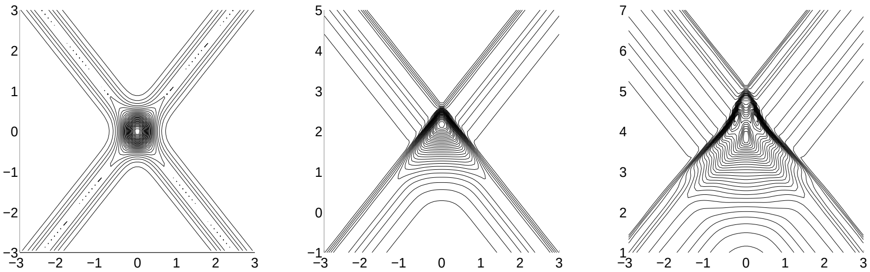
We first present results of a numerical calculation in which the initial conditions are

$$\begin{aligned} u(x,0) &= f_0(x \cdot v_1) v_1 + f_0(x \cdot v_2) v_2 \\ \rho(x,0) &= \left(\frac{a}{\sqrt{k\gamma}} \big(f_0(x \cdot v_1) + f_0(x \cdot v_2) \big) \right)^{\frac{1}{a}}. \end{aligned}$$

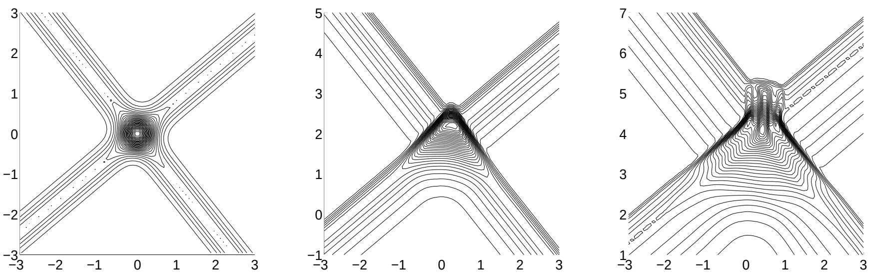
Here $f_0(s) = 2 + 3\exp(-5s^2)$, and there are two constant unit vectors v_1 and v_2. In the upper row of figures, the dot product

$$v_1 \cdot v_2 = -a.$$

In the lower row, v_2 was changed slightly.



Density contours for a case in which the angle is "correct".



Density contours for a case in which the plane waves begin "incorrectly" orthogonal.

The calculations were done in `clawpack` [2] [4]. Here $\gamma = 1.4$ and $k = 20\,000$. The times t in the figures are 0, 0.2667, and 0.5333, and the Burgers solution $f(s,t)$ remains smooth until approximately $t = 0.2$.

We next prove that there are such solutions.

Theorem. Let $v_1, \ldots, v_N$ be unit vectors in $\mathbb{R}^3$ for which the dot products

$$v_n \cdot v_m = -a, \qquad n \neq m$$

(N is 2, 3, or 4, depending on γ) and suppose that $f_n(s,t)$ are differentiable solutions to Burger's equation

$$f_t + (1+a) f f_s = 0, \qquad s \in \mathbb{R}, \quad 0 \leq t < T$$

Define

$$u(x,t) = \sum_{n=1}^{N} f_n(x \cdot v_n, t) v_n, \qquad \text{and } \rho = \Big(\frac{a}{\sqrt{k\gamma}} \sum_{n=1}^{N} f_n(x \cdot v_n, t) \Big)^{\frac{1}{a}}$$

Then u and ρ satisfy the polytropic Euler equations on this time interval, for as long as the sum of the f_n remains positive.

Proof. The calculation is simpler if we first transform the Euler system to symmetric form. Let w be the sound speed divided by a:

$$w = a^{-1}\sqrt{k\gamma\rho^{\gamma-1}}, \quad \rho = \left(\frac{a}{\sqrt{k\gamma}}w\right)^{\frac{1}{a}}.$$

The Euler system becomes

$$\begin{aligned} u_t + u\cdot\nabla u + aw\nabla w &= 0 \\ w_t + u\cdot\nabla w + aw\operatorname{div} u &= 0. \end{aligned}$$

This is can be written

$$q_t + A_1 q_{x_1} + A_2 q_{x_2} + A_3 q_{x_3} = 0, \qquad q = \begin{bmatrix} u \\ w \end{bmatrix}, \qquad A_i = u_i I + awL_i$$

where I is the 4×4 identity matrix and

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Set

$$q(x,t) = \sum_{n=1}^{N} f_n(x\cdot v_n, t)\begin{bmatrix} v_n \\ 1 \end{bmatrix}.$$

We abbreviate $\frac{\partial f_n}{\partial s}$ evaluated at $(x\cdot v_n, t)$ by f_{ns}. Component i of vector v_n is written v_{ni}. Also abbreviate $\frac{\partial f_n}{\partial t}(x\cdot v_n, t)$ by f_{nt}, and $f_n(x\cdot v_n, t)$ by f_n. Then

$$q_{x_i} = \sum_{n=1}^{N} f_{ns} v_{ni}\begin{bmatrix} v_n \\ 1 \end{bmatrix} \quad \text{and} \quad A_i q_{x_i} = \sum_{n=1}^{N} f_{ns}\left(u_i v_{ni} + awv_{ni}(e_i + v_{ni}e_4)\right)\begin{bmatrix} v_n \\ 1 \end{bmatrix}.$$

So

$$q_t + \sum_{i=1}^{3} A_i q_{x_i} = \sum_{n=1}^{N} f_{nt}\begin{bmatrix} v_n \\ 1 \end{bmatrix} + \sum_{n=1}^{N} f_{ns}\left(u\cdot v_n\begin{bmatrix} v_n \\ 1 \end{bmatrix} + awv_{ni}\begin{bmatrix} v_n \\ |v_n|^2 \end{bmatrix}\right)$$

$$= \sum_{n=1}^{N}\left(f_{nt} + f_{ns} q\cdot\begin{bmatrix} v_n \\ a \end{bmatrix}\right)\begin{bmatrix} v_n \\ 1 \end{bmatrix} = \sum_{n=1}^{N}\left(f_{nt} + f_{ns}(1+a)f_n\right)\begin{bmatrix} v_n \\ 1 \end{bmatrix} = 0.$$

This completes the proof.

Remark. Such configurations cannot generally live beyond the time when shocks develop in any of the f_n. For example, suppose a shock of speed σ develops in f_1, and $\gamma = 1.4$. The jump condition on density is $[\rho]\sigma = [\rho u] \cdot v_1$ or

$$\left[\Big(\frac{a}{\sqrt{k\gamma}}\sum_{n=1}^{N} f_n\Big)^{\frac{1}{a}}\right]\sigma = \left[\Big(\frac{a}{\sqrt{k\gamma}}\sum_{n=1}^{N} f_n\Big)^{\frac{1}{a}}(f_1 + af_2 + af_3)\right].$$

But this is not possible. Consider a line segment lying in a plane level set of f_3 and within the shock plane. Along this segment, f_2 will in general take a continuous range of values. Since $1/a = 5$, the jump condition is a polynomial identity in the values of f_2. This contradicts the fundamental theorem of algebra. The numerics suggest that perhaps some effect persists.

[1] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech., **1** (1948)

[2] `http://www.amath.washington.edu/~claw/`

[3] P. D. Lax, *Hyperbolic Partial Differential Equations.* Providence: American Mathematical Society, 2006.

[4] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems.* Cambridge University Press, 2002.