Notes for Math 323 Fall 2005
Euler PDE and Acoustic Waves

B. Terrell

July 14, 2005

1 Introduction

The purpose of these notes is to derive three PDEs which describe compressible flow of air in one dimension, say in a pipe. You might imagine pumping air into a bicycle tire through a hose, for example. The main assumptions are:

- There is no conduction of heat through the air. There is no friction of the air with the pipe, or any exchange of heat between the air and the pipe.
- You know about Newton's law $F = ma$, and about the idea of conservation of mass. And of course you know calculus.

Two of the three equations are known as the Euler equations for homentropic flow. You do not have to know what 'homentropic' means. Then having the Euler equations, we can further derive that certain approximate solutions ought to also give us solutions to the wave equation. These solutions are very familiar–they are known as *sound*. Thus we also get, almost free, a derivation of the wave equation.

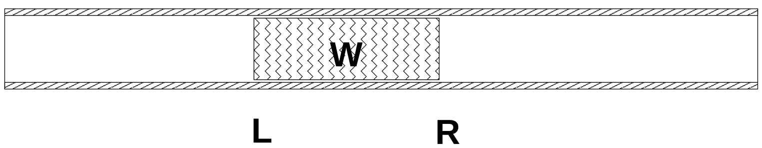
It is possible that you have derived the wave equation for vibrations of a string. If so, you will see that there are two very different situations giving rise to the same mathematics.

The method used in these notes is rather common in science. There are many assumptions behind every statement. So, to understand the derivation, you have to ask yourself at every step, Why should that be true? What is being assumed? More than anything else, it is this careful questioning which can lead to the greatest insights.

The biggest differences between these notes, and what you might find in a more advanced treatment, are that 1) we work in one dimension rather than three, and 2) we make some of that careful questioning explicit. But you can still find plenty of additional questions to ask.

2 The Euler equations

The pipe is full of air. The air velocity is $u(x,t)$ [m/sec], and density $\rho(x,t)$ [kg/m^3]. For air, the density is typically around 1.2 near the surface of the earth, but the velocity could vary over a large range. We can think of x and u positive toward the right.



We focus on a portion W of the air. The letters L and R in the figure refer to the left and right ends of the mass W. W is mathematically just a moving interval. The left coordinate L is moving at velocity u_L (or $u(L,t)$ if you prefer), and the right R at velocity u_R.
The area of the pipe is A. Besides u and ρ, there is the air pressure p [N/m^2]. It is very easy to mistake ρ and p typographically, but this won't happen to you because you are reading very slowly and thinking hard, right? The pressure provides forces $p_L A$ and $p_R A$ on the sides of W. The homentropic assumption is that the pressure is related to the density by $p = a\rho^\gamma$, where $\gamma = 1.4$ and a is constant throughout the flow. Aside from this thermodynamic fact, the rest of our derivation will consist of ordinary mechanics.

Newton: The rate of change of momentum of the matter in W is equal to the sum of applied forces.

$$\frac{d}{dt}\int_L^R \rho u A\,dx = -p_R A + p_L A$$

- Question 1. Is this really $F = ma$ or is it more like $\frac{d}{dt}(mv) = F$? When does the m in Newton's law have to be included inside the derivative? Of the quantities and expressions ρ, u, A, and $\int dx$, which constitute v and which m?
- Question 2. Why do we multiply pressure by area?
- Question 3. Why is there a minus sign on $p_R A$?

By the fundamental theorem of calculus, the right side in Newton's Law is equal to

$$-\int_W p_x A\,dx$$

We will use x and t subscripts to indicate partial derivatives, although L and R subscripts indicate evaluation. This won't confuse us, will it? For the left side, we need to remember that

$$\frac{d}{dt}\int_{g(t)}^{f(t)} h(x,t)\,dx = \int_{g(t)}^{f(t)} h_t(x,t)\,dx + h(t,f(t))f'(t) - h(t,g(t))g'(t)$$

- Exercise 1. Derive that from the Fundamental Theorem of Calculus and the Chain Rule.

So the left side of Newton's law is equal to

$$\int_L^R (\rho u)_t A\,dx + (\rho u)_R A u_R - (\rho u)_L A u_L$$

and again by the Fundamental Theorem this is

$$\int_L^R [(\rho u)_t + (\rho u^2)_x] A\,dx$$

So what we have derived is that for every moving interval of fluid $W = [L, R]$,

$$\int_L^R [(\rho u)_t + (\rho u^2)_x + a(\rho^\gamma)_x] A\,dx = 0$$

- Question 4. If a continuous function f has the property that $\int_a^b f(x)\,dx = 0$ for one interval $[a, b]$, can you deduce anything about f? How about when the integral is 0 for *every* interval? To what extent does it matter that our intervals are moving?

So we feel reasonably confident that Newton's Law can be expressed as

$$(\rho u)_t + (\rho u^2)_x + a(\rho^\gamma)_x = 0$$

We notice that we have two unknowns u and ρ. Probably one equation isn't going to be enough, so we look for another equation. Turning to conservation of mass, we figure the mass in W to be $\int_L^R \rho A\, dx$.
This doesn't change as the fluid moves because our set W is moving with the fluid. You could think of W as always the same set of molecules, but this isn't right because we aren't tracking molecular randomness. Anyway if the mass in W doesn't change then we have

$$\frac{d}{dt}\int_L^R \rho A\, dx = 0$$

Doing the same chain rule transformations as we did for Newton gives

$$\int_L^R [\rho_t + (\rho u)_x] A\, dx = 0$$

Again this is supposed to hold for all W, so the integrand must be zero.

- Exercise 2. Use $\rho_t + (\rho u)_x = 0$ to simplify the Newton PDE to $u_t + u u_x = -a\gamma\rho^{\gamma-2}\rho_x$ These two equations are the results of our work, the one-dimensional homentropic Euler equations.

- Exercise 3. We want to identify a physical meaning for the combination of terms $u_t + u u_x$ which occurs in the equation of motion. We can show that it is the acceleration of the particle which is passing through the point x at time t. Suppose a particle of fluid has position given by a function $x(t)$. Then we have two expressions for the velocity of this particle, $x'(t)$, and $u(x(t), t)$. These must be equal. Differentiate to show that the composition $(u_t + u u_x)(x(t), t)$ is equal to the acceleration of the air particle.

- Question 5. Can there be any acceleration at a point where $u_t = 0$?

3 Sound

Here we start from the Euler equations, and imagine a small disturbance superimposed over an ambient stillness.

- Question 6. Is it true that the ambient stillness $u = 0$, $\rho =$ (constant) is a solution to the Euler equations of Exercise 2?

We then look for approximate solutions of the form $u(x,t) = \epsilon v(x,t)$, $\rho(x,t) = \rho_1 + \epsilon w(x,t)$ where ϵ is supposed to be a small number. The constant ρ_1 could be taken to be a typical air density at sea level on earth.

- Exercise 4. Substitute our assumed u and ρ into the Euler equations and ignore everything containing ϵ^2 or higher powers. Show that you find for the ϵ^1 terms,

$$v_t = -a\gamma\rho_1^{\gamma-2} w_x$$
$$w_t + \rho_1 v_x = 0$$

Combining the two PDEs in Exercise 4, we have $w_{tt} = -\rho_1 v_{xt} = a\gamma\rho_1^{\gamma-1} w_{xx}$ or

$$w_{tt} = c^2 w_{xx}$$

where $c^2 = a\gamma\rho_1^{\gamma-1}$ So the pressure disturbance satisfies the wave equation.

- Question 7. What did we just now assume about the second derivatives? This subtlety is omitted routinely in scientific discussions. Are solutions to an approximate equation necessarily approximate solutions to the right equation? This point is also usually omitted.

- Question 8. Air pressure at sea level on the earth is about 10^5 [N/m^2]. Using this and previous information, estimate the value of the coefficient c^2 occuring in our wave equation. Then, recognizing that traveling waves such as $f(x - bt) + g(x + bt)$ satisfy a wave equation, what do we learn about the speed of sound? If lightning strikes at a distance of 3 football fields from you, how long before you hear it?

Finally, we have to ask:

- Question 9. Have we proved anything about the behavior of real air? What is the criterion for correctness in Physics? in Mathematics? Have these notes strictly conformed to either?

4 Appendix

You might find that you need to omit this appendix. It is also possible that you need to read it, in order to make sense out of the previous sections.

- Question 10. We decided to look for 2 equations for our 2 variables u and ρ. How reliable is the idea that one needs n equations in n unknowns before you can solve anything? Have you seen examples in linear algebra, $A\vec{x} = \vec{b}$, where two equations in three variables might have no solution? where five equations in three variables might have a unique solution? Think of the eigenvalue problem $A\vec{x} = \lambda\vec{x}$, which is nonlinear because we view both λ and $\vec{x}$ as unknowns. Suppose A is 8×8, how many variables are there? equations? solutions λ? solutions $\vec{x}$? There is some truth in this idea, but few guarantees.

- Question 11. In deriving the wave equation for sound in still air, we assumed $u = \epsilon v$, $\rho = \rho_1 + \epsilon w$. If we wanted instead to see how a disturbance propagates through a moving airstream, what might have to be changed about that assumption?

- Project 1. There are at least two ways to understand our earlier statement about the mass integral, that the mass in W is $\int_R^L \rho A\,dx$. You probably know that you can think of dx as a bit of length, $A\,dx$ as a bit of volume, $\rho A\,dx$ as a bit of mass, and the integral adds them. Another interesting approach is explained by P. Lax in his calculus book. Suppose we write $S(f, I)$ for the mass contained in interval $I = [a, b]$ when function f gives the density in I. S has two physically reasonable properties:

  1. If I is broken into nonoverlapping subintervals $I = I_1 \cup I_2 = [a, c] \cup [c, b]$, then $S(f, I) = S(f, I_1) + S(f, I_2)$
  2. If there are numbers m and M such that $m \le f(x) \le M$ then $m(b - a) \le S(f, [a, b]) \le M(b - a)$

  Under these conditions, you can actually prove that $S(f, [a, b]) = \int_a^b f(x)\,dx$. The project is to look up whatever definitions you need and figure out why this works. The idea applies to many applications other than mass.

You might be wondering whether we need to have another equation about conservation of energy. Ordinarily we would, but our homentropic assumption $p = a\rho^\gamma$ with no heating and no friction, has the consequence that all energy changes are accounted for already by Newton's law. For a simpler example, a frictionless mass on a spring has Newton's law $mx'' = -kx$. After multiplying by x' you can integrate once to get the conservation of energy $\frac{1}{2}m(x')^2 + \frac{1}{2}kx^2 = \text{constant}$. A similar thing happens here. If we were to allow heating of the air, then we would need another variable which could be taken to be temperature, and another equation.
There is plenty of room for misunderstanding about 'heating': the word heating means transfer of heat energy, as through the walls of the pipe. The temperature, which is not the same as the heating, does in fact change even without heating, because air obeys the ideal gas law $p = (\text{constant})\rho T$.

- Question 12. We showed that the density disturbance w satisfies the wave equation. You can show similarly that the velocity disturbance v satisfies the wave equation with the same coefficient c^2. What is the significance of the fact that the velocity and pressure disturbances satisfy the *same* wave equation? Specifically, is it physically plausible that these two disturbances might travel at different speeds?

- Project 2. Our derivations tracked the momentum and mass of a moving portion of fluid. Some people prefer instead to track the momentum and mass inside a fixed interval, accounting for stuff going in and out. Do you think one of these approaches might be more correct physically than the other? The project is to redo the Euler equation derivation by considering a nonmoving interval $N = [a, b]$ instead of the moving W. The mass conservation is easiest: The rate of change of the mass which is currently contained in N is

  $$\frac{d}{dt}\int_a^b \rho A\, dx$$

  and this ought to be equal to the rate in at b plus the rate in at a:

  $$= -\rho_b A u_b + \rho_a A u_a$$

  Of course this leads to the same mass PDE as before. Try to redo the Newton law in this context.