# The isentropic Euler system admits some plane wave superpositions

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## 1. Main Result

These velocities add, and pressures combine nonlinearly.

$$A_1 = \begin{bmatrix} u_1 & 0 & 0 & aw \\ 0 & u_1 & 0 & 0 \\ 0 & 0 & u_1 & 0 \\ aw & 0 & 0 & u_1 \end{bmatrix},\ A_2 = \begin{bmatrix} u_2 & 0 & 0 & 0 \\ 0 & u_2 & 0 & aw \\ 0 & 0 & u_2 & 0 \\ 0 & aw & 0 & u_2 \end{bmatrix},\ A_3 = \begin{bmatrix} u_3 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 \\ 0 & 0 & u_3 & aw \\ 0 & 0 & aw & u_3 \end{bmatrix}$$

Isentropic Euler equations in symmetric [3] form

$$q_t + A_1 q_{x_1} + A_2 q_{x_2} + A_3 q_{x_3} = 0$$

where $q = (u_1, u_2, u_3, w)$ is a vector of velocities together with $w = a^{-1}\sqrt{\gamma p/\rho}$, proportional to the sound speed.

Here $p = k\rho^\gamma$ with $k$ constant, $1 < \gamma < 3$, and $a = \frac{\gamma-1}{2}$.

THEOREM

There are solutions $q(x,t) = \sum_{j=1}^{N} f_j(x \cdot v_j, t) \begin{bmatrix} v_j \\ 1 \end{bmatrix}$ i.e.

$$u(x,t) = \sum_{j=1}^{N} f_j(x \cdot v_j, t) v_j,$$

and

$$\rho = \left( \frac{a}{\sqrt{k\gamma}} \sum_{j=1}^{N} f_j(x \cdot v_j, t) \right)^{\frac{1}{a}}.$$

Each $f_j(s,t)$ is a smooth inviscid Burgers solution

$$f_t + (1+a) f f_s = 0, \qquad s \in \mathbb{R}, \quad 0 \le t < T$$

and $v_j$ are constant unit vectors with

$$v_i \cdot v_j = -a, \qquad i \neq j.$$

The solutions exist for as long as the $f_j$ remain smooth and $\sum f_j > 0$.

| $\mathbb{R}^d$ | $1 < \gamma < \frac{5}{3}$ | $\gamma = \frac{5}{3}$ | $\frac{5}{3} < \gamma < 2$ | $\gamma = 2$ | $2 < \gamma < 3$ |
|---|---|---|---|---|---|
| $\mathbb{R}^2$ | 2 | 2 | 2 | 3 | 2 |
| $\mathbb{R}^3$ | 3 | 4 | 3 | 3 | 2 |

The table shows the number $N$ of vectors $v_k$ available for various $d$ and $\gamma$.

For details see
`http://www.math.cornell.edu`
`/~bterrell/eulersuper.pdf`

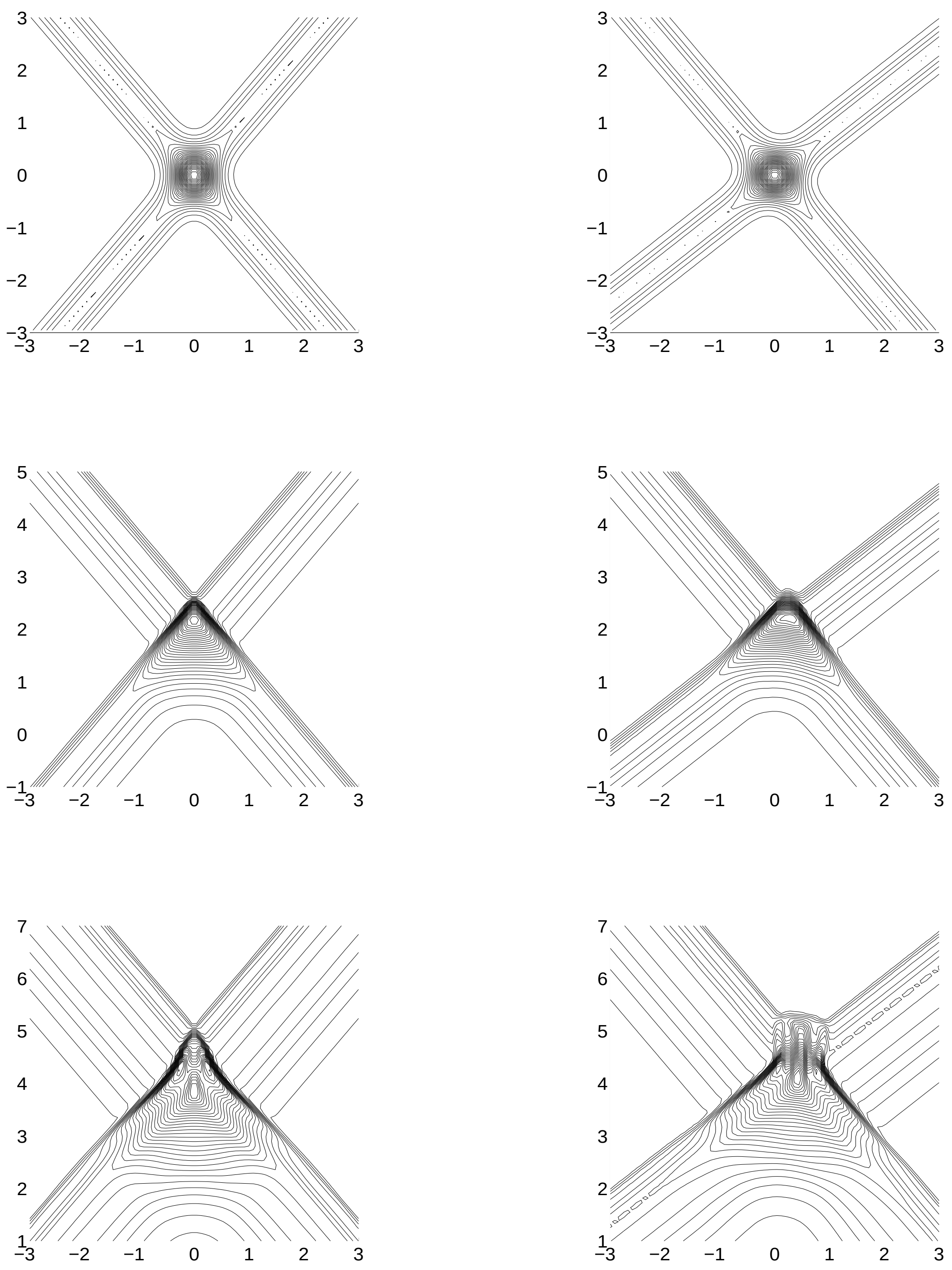
## 2. Special cases

1) $\gamma = 2$, *shallow water model*, three vectors $v_k$ coplanar at $120$ degrees
2) $\gamma = \frac{5}{3}$, *monatomic gas*, four vectors $v_k$ having the symmetry of a regular tetrahedron.

## 3. A computation

Illustration using a `clawpack`[2], [4] calculation. Density contours shown, with time increasing downward.

$\gamma = 1.4$, $k = 20\,000$, $f_1(s,0) = f_2(s,0) = 2 + 3\exp(-5s^2)$, $f_3 = 0$. The solutions remain smooth almost until the time of the second figures.

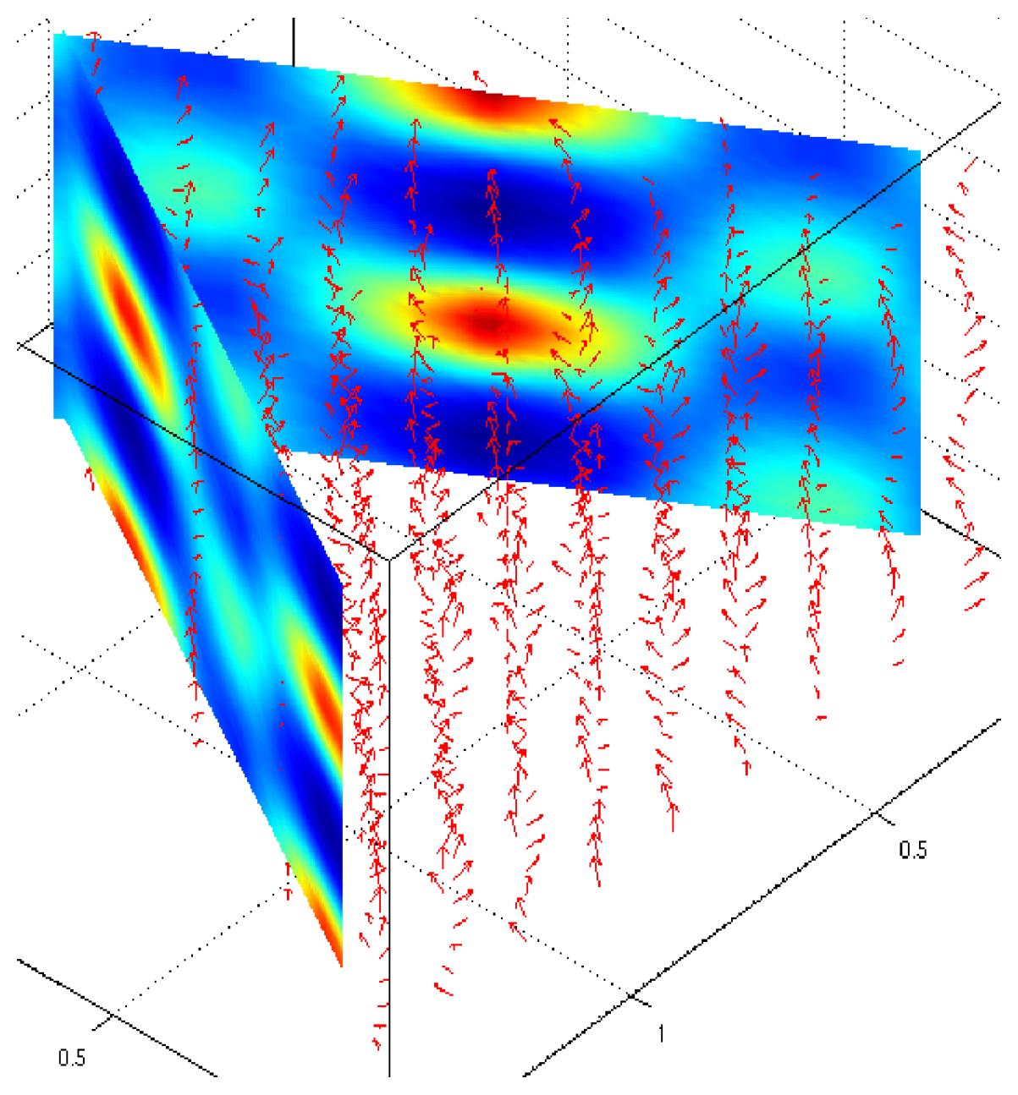


Left: angle is “correct”.
Right: angle is “incorrectly” orthogonal.

## 4. Example in three dimensions

A flow in a triangular duct, tangential to the boundary.
$v_1 = (\sqrt{3},1,1)/\sqrt{5}$, $v_2 = (-\sqrt{3},1,1)/\sqrt{5}$, $v_3 = (0,-2,1)/\sqrt{5}$. $\gamma = 1.4$

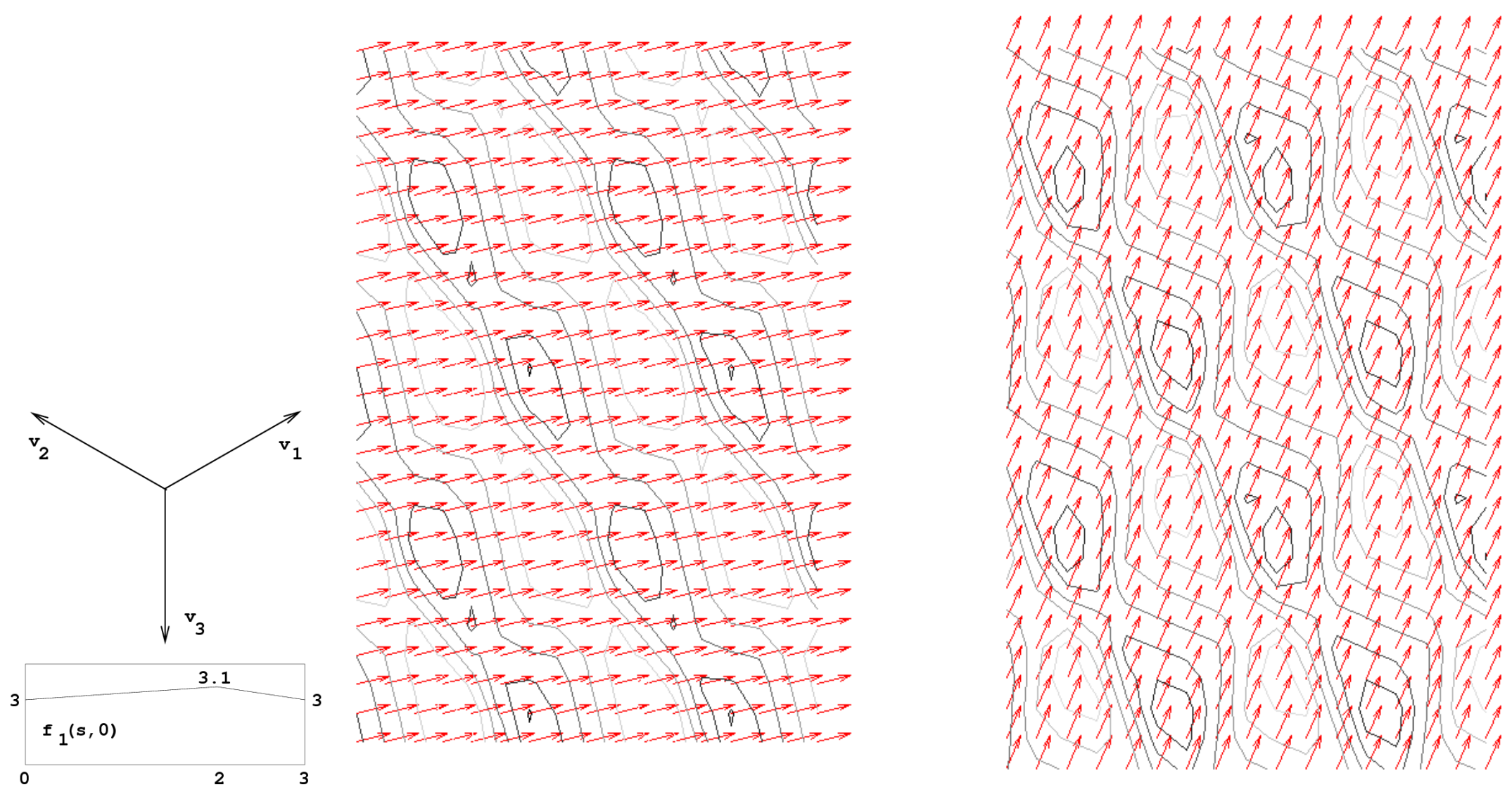
$f_1(s,0) = f_2(s,0) = f_3(s,0) = f(s)$ has period $p$, and the duct sides have width $\sqrt{5}p/3$.
Here $f(s) = 2 + 0.5\cos(3\pi s)$.



## 5. Example in two dimensions

An exact “shallow water” model, $\gamma = 2$, $g = 9.8$, $k = \frac{g}{2}$.

Left: $f_1(s,0)$ is graphed, period 3, and the $v_j$ indicated. The $f_3 = 0$.
Center: $f_2(s,t) = -0.4 f_1(s, -0.4t)$, $t = 0.5$, depth $\sim 0.09$, speed $\sim 3.8$.
Right: $f_2(s,t) = 0.6 f_1(s, 0.6t)$, $t = 0.5$, depth $\sim 0.6$, speed $\sim 2.7$.



## References

[1] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech., 1 (1948) pp. 171-179

[2] `http://www.amath.washington.edu/~claw/`

[3] P. D. Lax, *Hyperbolic Partial Differential Equations*, Providence, RI, American Mathematical Society, 2006.

[4] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002.