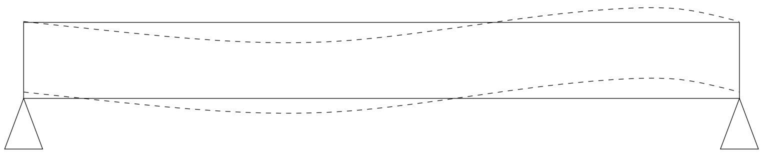
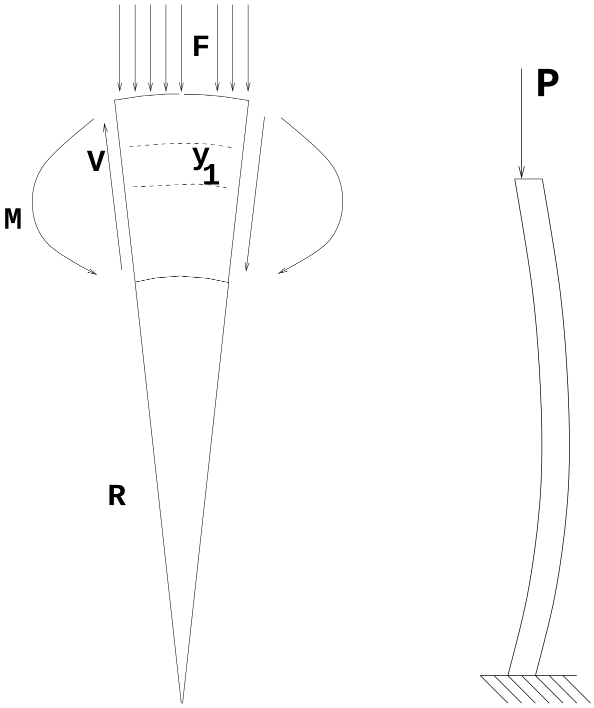
A Description of the differential equations for Beams and Columns



A beam is a bar loaded transverse to the axis, while a column is loaded along the axis. Of course the orientation doesn't matter even though we think of a column as being vertical,



For either case suppose x axis left to right gives position along the bar, and we have a segment of the bar as shown with shear force V [Newtons] and bending moment M [Newton-meters] as functions of x. Also F [N/m] is an applied loading. Let $y(x)$ be the downward displacement of the beam measured to the centerline say, and R the radius of curvature at each point, also measured to the centerline. In mechanics class you will use 'centroid' which is more correct than centerline.

Geometric fact, not included in most calculus courses: $y'' = \frac{1}{R}$. They do tell you about concave up and down (*up* is shown for the segment!) but this quantifies it. You can check this fact by writing a function whose graph is a circle, and verify the second derivative.

The hardest part of the analysis is to see that y'' is also proportional to M. Here is how that works.

For the bending shown, the top portion of the beam must stretch, and the bottom compress. Try it with an eraser. Mechanical fact: steel has the property that for small stretching and compressing, the stress σ =(force/area) and the strain ε =(length change)/(original length) are related by Young's modulus $E = 210 \times 10^9$ Newtons/(square meter) as

$$\sigma = E\varepsilon$$

For aluminum and granite E is about 1/3 as much.

Next geometric fact: let y_1 measure the distance from the centerline of the beam, away from the center of curvature. So $y_1 = 0$ at the centerline, $y_1 = +2$ at the top of the beam if the whole size of the beam is 4 units top to bottom, etc. Then the geometric fact is that the length at level y_1 changes by the factor $(1 + \frac{y_1}{R})$. Exercise: check that out. Thus $\varepsilon = \frac{y_1}{R}$.

So, the bending moment at a particular cross section must cause the length changes, and we have stress

$$\sigma(y_1) = E\frac{y_1}{R}$$

pulling or pushing normal to the cross section. Then the moment is the sum of force times moment arm,

$$M = \iint_{\mathrm{cross-section}} \sigma(y_1) y_1 \, dA = E \iint_{\mathrm{cross-section}} y_1^2 \, dA\left(\frac{1}{R}\right) = EI\left(\frac{1}{R}\right) = EIy''$$

This concludes the hardest part.

For example a $q \times q$ square cross section beam has moment of inertia $I = \int_{-q/2}^{q/2}\int_{-q/2}^{q/2} y_1^2 \, dy_1 dy_2 = q^4/12$.

The easier parts are to see that M' is proportional to the shear V, and that V' is proportional to F, the downward load [N/m] applied to the beam.

Why: For M' sum moments about the left end point of the segment, giving $-M(x) + M(x + \Delta x) + (\Delta x)V(x + \Delta x) = 0$. For V' sum downward forces on the segment, giving $-V(x) + V(x + \Delta x) + F(x)(\Delta x) = 0$. Then divide by Δx and take limits as $\Delta x \to 0$.

Putting all that together we summarize beams: (sign conventions may differ in various texts)

$$EIy'' = M, \quad EIy''' = -V, \quad EIy'''' = F$$

These are differential equations, and in addition they tell us how to interpret various boundary conditions. For example a cantilever beam embedded solidly into concrete at one end probably has $y = y' = 0$ at the concrete end, and $y'' = y''' = 0$ at the free end.

Two more applications:

For a column which is deflected into a bent shape $y(x)$ by an axial force P, you get a bending moment $y(x)P$ at each cross section, and taking account of signs the DE becomes

$$EIy'' = -Py$$

Finally we mention vibrating beams: Suppose y depends on x and t and that there is no applied load F. If the mass of the beam is ρ [kg/m] then you find a PDE

$$EIy_{xxxx} = -\rho y_{tt}$$

for the vibrations.