det

The determinant determines whether the matrix is invertible.

1 2 by 2

You know that for 2 by 2 matrices, the determinant is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and it determines whether the matrix is invertible. It is used in the formula for the inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It also tells geometric effects, for example if $\det(A) = -3$ then A expands areas by a factor of 3 and reverses the orientation. The determinant was discovered when someone (like you, you could try this) solved a linear system symbolically and realized that $(ad - bc)$ always shows up in the denominator.

2 3 by 3

If you solve the system

$$\begin{bmatrix} a & b & c \\ u & v & w \\ r & s & t \end{bmatrix} \vec{x} = \vec{b}$$

symbolically then you find the expression

$$avt + bwr + cus - aws - but - cvr$$

in the denominators of the x_k. That then is defined to be the determinant of the 3 by 3 matrix. It is so complicated that people have found various ways to express it, such as:

$$\det \begin{bmatrix} a & b & c \\ u & v & w \\ r & s & t \end{bmatrix} = a \det \begin{bmatrix} v & w \\ s & t \end{bmatrix} - b \det \begin{bmatrix} u & w \\ r & t \end{bmatrix} + c \det \begin{bmatrix} u & v \\ r & s \end{bmatrix}$$

Example: $\det \begin{bmatrix} 3 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 3$. This matrix stretches or compresses the three axes by various factors, and it multiplies volumns by 3. The matrix is invertible. The determinant of the inverse is 1/3.

3 any size

You can find the determinant of any square matrix iteratively as in this example, note the alternating signs:

$$\det \begin{bmatrix} a & b & c & d \\ 0 & 1 & 2 & 0 \\ -2 & 4 & 6 & 7 \\ -3 & 0 & 5 & 8 \end{bmatrix} = a \det \begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & 7 \\ 0 & 5 & 8 \end{bmatrix}$$

$$-b \det \begin{bmatrix} 0 & 2 & 0 \\ -2 & 6 & 7 \\ -3 & 5 & 8 \end{bmatrix} + c \det \begin{bmatrix} 0 & 1 & 0 \\ -2 & 4 & 7 \\ -3 & 0 & 8 \end{bmatrix} - d \det \begin{bmatrix} 0 & 1 & 2 \\ -2 & 4 & 6 \\ -3 & 0 & 5 \end{bmatrix}$$

Then do the 3 by 3's, etc, like:

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & 7 \\ 0 & 5 & 8 \end{bmatrix} = 1 \det \begin{bmatrix} 6 & 7 \\ 5 & 8 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 7 \\ 0 & 8 \end{bmatrix} + 0 \det \begin{bmatrix} 4 & 6 \\ 0 & 5 \end{bmatrix}$$

$$= 1(48 - 35) - 2(32 - 0) + 0(20 - 0) = -51$$

4 products

Another generally useful fact: $\det(AB) = \det(A)\,\det(B)$. For example, if A is invertible then

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\,\det(A^{-1})$$

so $\det(A^{-1}) = 1/\det(A)$.

A second application of that product rule:

Cramer's Rule: M. Cramer cleverly noticed for example that

$$\text{If} \quad \begin{bmatrix} a & b & c \\ u & v & w \\ r & s & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} P \\ D \\ Q \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} a & b & c \\ u & v & w \\ r & s & t \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P & b & c \\ D & v & w \\ Q & s & t \end{bmatrix}$$

and taking determinants you get a formula for the answer:

$$\det(A)x_1 = \det \begin{bmatrix} P & b & c \\ D & v & w \\ Q & s & t \end{bmatrix}$$

Similary for x_2 and x_3, and for any size system. Cramer's rule is a theoretical tool these days, not computationally efficient.