

## A brief on the complex linear spaces $\mathbb{C}^n$

For background on the complex numbers  $\mathbb{C}$ , including a motivated definition of  $e^{\text{complex}}$ , see my Notes on Differential Equations at

<http://www.math.cornell.edu/~bterrell/dn.pdf>

Then  $\mathbb{C}^n$  consists of all column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where each  $x_k$  is a complex number. This includes the possibility that all the  $x_k$  are real, so  $\mathbb{C}^n$  contains all the real vectors of  $\mathbb{R}^n$ . And  $\mathbb{C}^n$  is closed under addition and multiplication by real numbers. For example

$$y = \begin{bmatrix} i \\ 2 \\ \frac{3}{5} - 7i \end{bmatrix}, \quad z = \begin{bmatrix} 6 + 2.5i \\ 2 \\ \frac{3}{i} - 7 \end{bmatrix}, \quad y - 2z = \begin{bmatrix} -12 - 4i \\ -2 \\ \frac{3}{5} - 14 - i \end{bmatrix}$$

But there are differences from  $\mathbb{R}^n$ :

- You can multiply by complex scalars, not just real ones.
- You cannot find a useful vector length by squaring: if  $u = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$  then  $u_1^2 + u_2^2 = 0$ . But draw a picture to see that  $|1 \pm i| = \sqrt{2}$ .
- There is complex conjugation, for example  $\bar{u} = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix}$
- The dot product is different: First note that if you form the ordinary "dot" of  $\bar{u}$  and  $u$  you get  $(1-i)(1+i) + (1+i)(1-i) = 2 + 2 = 4$ , not 0. Convince yourself: This is what the squared length of  $u$  ought to be.
- So: we define  $\langle x, y \rangle = \sum \bar{x}_k y_k$  and use this for our official dot product in  $\mathbb{C}^n$ .  $\langle x, y \rangle$  is a complex number, but when  $y = x$  it is a positive real number, except for the case of  $x = 0$ .
- We define  $|x| = \sqrt{\langle x, x \rangle}$ .

Everything we know about linear algebra in  $\mathbb{R}^n$  with real scalars goes over to  $\mathbb{C}^n$  with complex scalars, as long as we remember to use the correct dot and length. For example the expansion of  $x$  in an orthonormal basis  $\{v_k\}$  is  $x = \sum_k \langle v_k, x \rangle v_k$ .

$\mathbb{C}^n$  is needed in quantum mechanics, electrical engineering, and is good fun too.

Note:  $\mathbb{C}^1$  looks like  $\mathbb{R}^2$ , but the algebra is different:  $\mathbb{R}^2$  has dimension 2 over the real scalars, but  $\mathbb{C}^1$  has dimension 1 over the complex scalars!