Solutions to Final Exam
Introduction to Topology, Math 453
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A qualification – I wrote these up in a hurry so don't take these solutions as the model for a perfect write-up, take this more as a guideline for how one could approach each problem. Have a great break, everyone!

Problem 1 (15 points) Let X denote the diadic rationals (those rational numbers which can be expressed in the form $\frac{m}{2^n}$ for some integers m, n. Let Y denote the remaining rationals. Let Z denote the irrationals. Note that each of the sets X, Y, and Z is dense in $\mathbb{R}$. We consider a finer topology on $\mathbb{R}$ than the usual: in addition to the usual open sets, we add X, Y, and sets of the form:

$$\{z\} \cup \{w \in X \cup Y \mid |w - z| < \delta\}$$

where $z \in Z$ and $\delta > 0$.

(a) (5 points) Show that as subspaces of $\mathbb{R}$ with this topology, X and Y are totally disconnected.

You can answer both parts of (a) and (b) in this question with the following argument. Let $K \subset \mathbb{R}$ be a subset such that $\mathbb{R} - K$ is dense in $\mathbb{R}$. We will show that K is totally disconnected. Let L be a subset of K containing at least two points, say $a < b$, without loss of generality. Since $\mathbb{R} - K$ is dense in $\mathbb{R}$, there exists a $c \in \mathbb{R} - K$ such that $a < c < b$. Then $K \cap (-\infty, c)$, $K \cap (c, \infty)$ form a separation of K. Hence the only connected subspaces of K are one-point sets.

(b) (5 points) Show that $X \cup Z$ and $Y \cup Z$ are both also totally disconnected.

See (a) above.

(c) (5 points) Show that $\mathbb{R}$ with this topology is connected.

This one caused some more grief than (a) and (b), understandably. There are a few different approaches to this. Here I will give one which is similar to the proof that $\mathbb{R}$ with the usual topology is connected. First, note that a basis for the new topology on $\mathbb{R}$ (which we didn't need for the previous parts of this problem) is given by sets of the following four types:

1. The usual intervals (a, b)
2. $(a, b) \cap X$
3. $(a, b) \cap Y$
4. $z_{ab} = \{z\} \cup ((a, b) \cap \mathbb{Q})$, where $z \in (a, b)$ is irrational

Suppose that U, V is a separation of $\mathbb{R}$ (we will understand throughout that we mean $\mathbb{R}$ with the new topology). Without loss of generality, we can assume that there exist $u \in U$ and $v \in V$ such that $u < v$. Then we let p be the least upper bound of the set $U_v = \{u \in U | u < v\}$.

If $p \in U$, then since $p \neq v$ and since U is open, we can find a basis element B_p such that $p \in B_p \subset U$, and in any case we get a contradiction of p being the least upper bound of U_v.

If $p \in V$, as above we can find a basis element B_p such that $p \in B_p \subset V$. Suppose now that $p \in X$. Then we may assume that B_p is of the 2nd type, that is, $B_p = (a, b) \cap X$ for some $a, b \in \mathbb{R}$. We would like to contradict the fact that p is a least upper bound for U_v again. The danger is that the interval (a, b) may contain points of U arbitrarily close to p; we will show this can't happen.

Suppose that $z \in (a, b)$. Then if $z \in U$, since U is open, there exists a basis element of the fourth type containing z, contained entirely in U. However, such a basis element necessarily intersects B_p, and so z must be an element of V. Thus V must contain a basis element of the form z_{cd} for every $z \in (a, b)$, which implies that $(a, b) \cap Y$ is also contained in V. So we have $p \in (a, b) \subset V$, which clearly contradicts p being a least upper bound for U_v.

The case $p \in Y$ is identical and the case $p \in Z$ is a modified, easier version.

Problem 2 (20 points) There are two fundamental theorems of topology which are deceptively simple statements concerning *simple closed curves* (embeddings of S^1) in the plane.

The Jordan Curve Theorem. If J is a simple closed curve in the plane $\mathbb{R}^2$, then $\mathbb{R}^2 - J$ has two connected components, and J is the boundary of each.

The Schönflies Theorem: If J is a simple closed curve in $\mathbb{R}^2$, then one of the connected components of $\mathbb{R}^2 - J$ is homeomorphic to the open unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$.

(a) (5 points) Formulate a version of the Schönflies Theorem with S^2 replacing $\mathbb{R}^2$ (but you do not need to prove it).

The Schönflies Theorem, S^2 version: If J is a simple closed curve in S^2, then each of the connected components of $S^2 - J$ is homeomorphic to the open unit disk D^2.

(b) (8 points) Show that the Jordan curve theorem implies the following theorem and vice-versa:

Theorem: If L is a closed subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}^1$, then $\mathbb{R}^2 - L$ has two components and L is the boundary of each.

Before we begin, a few preliminary remarks. Recall that S^2 is the 1-point compactification of $\mathbb{R}^2$. Thus we can always view $\mathbb{R}^2$ as being contained in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ as a subspace. Also, given any point $p \in S^2$, stereographic projection gives us a homeomorphism from $S^2 - p$ to $\mathbb{R}^2$.

Also, I draw your attention to Lemma 61.1 of your book, which basically says that under stereographic projection, the connected components of S^2 minus a simple closed curve go where you'd expect them to go in $\mathbb{R}^2$. We will use this implicitly to derive the desired result.

Finally, the term "closed" in simple closed curves simply refers to the fact that it's a loop, i.e., it "closes up". However, a simple closed curve in either $\mathbb{R}^2$ or S^2 is also a closed subset in the usual sense.

Now, assume the Jordan Curve Theorem (JCT). Let L be a closed subset of $\mathbb{R}^2$ homeomorphic to $\mathbb{R}$. The assumption that L is closed ensures that it "goes off to infinity" at both ends, e.g., it doesn't spiral in on any point in $\mathbb{R}^2$ (which would therefore be a limit point). Consider the 1-point compactification $\mathbb{R}^2 \cup \{\infty\} \cong S^2$; in this way we will think of our original $\mathbb{R}^2$ as a subset of S^2. Then $J = L \cup \{\infty\} \cong S^1$, by uniqueness of 1-point compactification, and J is a simple closed curve in S^2. Now choose any point $p \in S^2$ such that $p \notin J$, and stereographically project back to $\mathbb{R}^2$ using p as the new "north pole". The image of J under this stereographic projection (which we still call J by abuse of notation) is a simple closed curve in the plane. We can now apply the JCT to say that there are two connected components in the complement of J in the plane, and that J is the boundary of each. Reversing this 2-step process (and citing Lemma 61.1), we can conclude that the complement of L in the original plane consists of two components, and that L is the boundary of each.

A careful analysis of the above argument shows that each step is really an "if and only if", and so we also get the reverse implication for free.

(c) (7 points) Use the Schönflies Theorem (and your version of it for S^2) to prove the Annulus Theorem:

Annulus Theorem: The closure of the region between two disjoint simple closed curves in S^2 is homeomorphic to the annulus $S^1 \times [0, 1]$.

Let J_1, J_2 be two disjoint simple closed curves in S^2. By the Schönflies Theorem, $S^2 = J_1$ has precisely 2 components, C_1 and C_2, both homeomorphic to an open disk. Since J_2 is connected and disjoint from J_1, it must lie entirely in one of them, say in C_2. But $C_2 \cong D^2 \cong \mathbb{R}^2$, so by Schönflies (and JCT, implicitly), $C_2 - J_2$ has two components, one of which is a disk. Thus the region in question is homeomorphic to $\mathbb{R}^2$ minus a disk, or a disk minus a disk, however you prefer to think of it.

Contrary to popular belief, this does NOT automatically give us the desired result, in fact, this is where the real work is. How do we know the "inner" disk isn't embedded in the "ambient" disk in some totally bizarre way? For example, one needs some extra hypotheses to generalize the Schoönflies and annulus theorem in higher dimensions, thanks to "Alexander's horned sphere", which is an embedding of S^2 in S^3 which is so messed up that one of its complementary components is not even simply connected, let alone an open ball.

There are a few different ways to prove the result; I'll outline one approach here. We have J_1 bounding a disk, which we might as well take to be the actual unit disk. The disk contains the simple closed curve J_2, possibly all twisted up, but bounding a disk. Now draw 2 disjoint arcs connecting J_2 to J_1. (Why can we do this? Consider the function $(x, y) \mapsto x$ for points $(x, y) \in J_2$. This function attains both a min and a max, since $J_2 \cong S^1$ is compact. Use a min and a max point to draw your 2 arcs.)

Using the 2 arcs, we can find two simple closed curves in the disk, each made up of a segment of J_1, a segment of J_2, and the two arcs. Viewing the disk as being contained in $\mathbb{R}^2$, we can apply Schoönflies again and say that each bounds a disk. Thus the region in question is the union of two disks, with two disjoint arcs identified on the boundary of each. We can map each of the two disks into appropriate "halves" of our annulus $S^1 \times I$ homeomorphically. Since these maps will agree on the identified boundary arcs, we obtain a homeomorphism of the entire region into the annulus. See figures below.

One final remark – many of you attempted a proof relying on a variation of the following incorrect statement: if $A_1 \subset X_1$ and $A_2 \subset X_2$, and $A_1 \cong A_2$, and $X_1 \cong X_2$, then $X_1 - A_1 \cong X_2 - A_2$. If this were true, then all knot theorists would be out of business. Think of the "unknot", i.e., the standard embedding of S^1 in S^3, whose complement turns out to be a solid torus (minus its boundary), versus any other "knotted" knot sitting in 3-space, whose complement will be a lot more complicated, though proving this requires more technology.

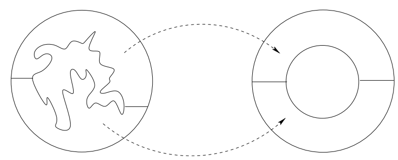

Figure 1: Two disks joined along two arcs in their boundaries.

Problem 3 (15 points total)

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two coverings of a space B. We define a *homomorphism* of (E_1, p_1) into (E_2, p_2) to be a continuous map $\phi : E_1 \to E_2$ such that $p_2 \circ \phi = p_1$. Show that if ϕ is a homomorphism of (E_1, p_1) into (E_2, p_2), then $\phi : E_1 \to E_2$ is itself a covering map.

We begin by recalling that we assume all spaces to be path-connected; this will be crucial for proving surjectivity of ϕ.

Let $b \in B$. We first show that b has a neighborhood U evenly covered by both covering maps. Since p_1, p_2 are covering maps, we can choose neighborhoods U_1, U_2 of b which are evenly covered by p_1, p_2, respectively. Then $U = U_1 \cap U_2$ is evenly covered by both maps, at least, as defined by your book. (In class, I added the restriction that to be "evenly covered" a neighborhood should also be path-connected, so in this case just take U to be the path component of $U_1 \cap U_2$ containing b. I'll forget about this detail from now on, since most of you did too and I decided not to take off points for it.)

Next we prove that ϕ is surjective. Let $y \in E_2$. Choose a basepoint $b_0 \in B$, and let $x_0 \in p_1^{-1}(b_0)$. Let $y_0 = \phi(x_0)$. Choose a path f in E_2 with $f(0) = y_0$ and $f(1) = y$. Let $g = p_2 f$; then g is a path in B with $g(0) = b_0$. By path-lifting, there is a unique lifting of g to a path h in E_1 with $h(0) = x_0$. Let $x = h(1)$. Note that ϕh is a path in E_2 with initial point y_0, and that $p_2 \phi h = p_1 h = g$. Also note that f is a path in E_2 with initial point y_0 which also projects to g under p_2. Thus $\phi h = f$ by uniqueness of path-lifting, and we have $\phi(x) = \phi(h(1)) = f(1) = y$.

To finish off the proof that ϕ is a covering map, choose U to be a neighborhood of $p_2(y)$ evenly covered by both p_1 and p_2. Let W be the component of $p_2^{-1}(U)$ containing y. Consider the collection the collection $\{V_\alpha\}$ of path components of $\phi^{-1}(W)$. This is a subcollection of the path components of $p_1^{-1}(U)$, and so for each α we have $\phi|_{V_\alpha} = (p_2|_W)^{-1} \circ p_1|_{V_\alpha}$. Since both of the restriction maps on the right-hand-side are homeomorphisms, so is ϕ.

Problem 4 (25 points total) Consider the topological space X consisting of two distinct concentric circles in the plane. Let C_1 denote the inner circle and C_2 denote the outer circle. We put a topology on X by taking as a subbasis the collection of all single-point sets in C_2 together with all open "intervals" on C_1 each together with the radial projection of all but its midpoint on C_2, as indicated in the figure.

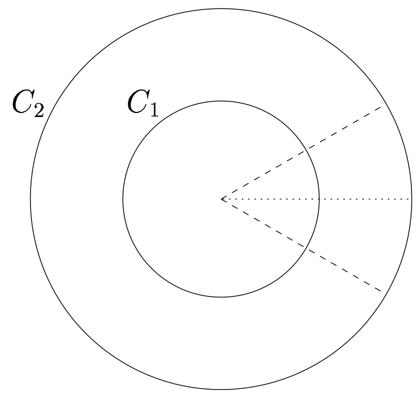

Figure 2: One type of subbasis element.

Decide which of the following terms apply to the space X: connected, compact, Hausdorff, regular, normal, 1st countable, 2nd countable, separable, Lindelöf, metrizable. In each case, give a short proof of why or why not the term applies. Be efficient!

Okay, here's the fastest way I know how to do this:

X **is Hausdorff:** Let $x, y \in X$. If $x, y \in C_2$, then $\{x\}, \{y\}$ are the disjoint open sets we're looking for. If $x, y \in C_1$, then we may clearly choose distinct subbasis elements of the second type by taking a small enough "interval" (you may cite Hausdorff properties of S^1, if you wish, noting that C_1 is homeomorphic to S^1 in the subspace topology, as long as you then translate this into the required open sets of X). If $x \in C_1$, say, and $y \in C_2$, then whether or not y is the radial projection of x, we can always choose a subbasis interval of the second kind which includes x but not y, then take $\{y\}$ as the second open set.

X **is not connected:** Since X is Hausdorff, single point sets are closed. Any single point set in C_2 is also open. Thus we have a proper, nonempty subspace of X (in fact, uncountably many of them) which is both open and closed, and therefore X is not connected. (Actually, the connected components of X turn out to be C_1 and every single point set in C_2.)

X **is compact:** Take any cover of X, then it is also a cover of C_1. As previously noted, C_1 in the subspace topology is just S^1, which is compact. Therefore, we can take a finite subcover of our original cover and cover C_1. Returning to X, we note that since any open set in X containing C_1 can miss only a finite number of points of C_2, so now we just throw in one open set containing each of them from the original cover. As one of you said, "finite + finite = finite".

X **is Lindelof:** Compactness implies the Lindelof condition immediately from the definitions.

X **is normal:** I stated a theorem in class which is proved in Section 32 of your book, that every compact Hausdorff space is normal.

X **is regular:** Every normal space is regular.

(My sympathies to those of you who ended up having to prove normal or regular directly – it's no fun, but I was impressed with your determination!)

X **is not separable:** Suppose D is dense in X. Then $\bar{D} = X$. Let $x \in C_2$. Since every open set containing x must intersect D (because the closure of D is all of X), in particular, the single point set $\{x\}$ intersects D, i.e., $x \in D$. Thus D contains all of C_2 and must therefore be uncountable.

X **is not 2nd countable:** Theorem 30.3 of your book says it all: 2nd countable implies separable, so by the above, X can't be 2nd countable.

X **is not metrizable:** Thanks to a well-loved homework problem (HW10, p. 194, Problem 5), we know that a metrizable Lindelof space is 2nd countable. We already know X is Lindelof, so if it were to be metrizable as well, this would contradict the fact that we know X is not 2nd countable.

X **is 1st countable:** Let $x \in C_2$, then $\{x\}$ is the countable basis at x we require. If $x \in C_1$, then just take any countable, nested sequence of subbasis elements of the second type (say, using rational endpoints), centered at x.

Problem 5 (25 points total)

Let A be a subset of a topological space X. We say that A is a *deformation retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ (the map r is called a *retraction*) and a homotopy $f : X \times I \to X$ such that $f(x, 0) = x$ and $f(x, 1) = r(x)$ for all $x \in X$ and $f(a, t) = a$ for all $a \in A$ and all $t \in I = [0, 1]$.

(a) (5 points) Prove that if A is a deformation retract of X, then the inclusion map $i : A \to X$ induces an isomorphism of the corresponding fundamental groups.

Choose a basepoint $a_0 \in A$. We would like to show that the homomorphism induced by inclusion:

$$i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$$

is in fact an isomorphism.

Consider the map $r \circ i : A \to A$. It is clear that $r \circ i$ is the identity map on A. Hence (by Theorem 52.4) $r_* \circ i_* = (r \circ i)_*$ is the identity homomorphism on $\pi_1(A, a_0)$.

Now consider $i \circ r : X \to X$. Since A is a deformation retract of X, there exists a homotopy f between $i \circ r$ and id_X, the identity map on X. The homotopy f fixes every point of A at every stage; in particular, the basepoint a_0 remains fixed throughout. Now apply Lemma 58.1 to conclude that $i_* \circ r_* = (i \circ r)_*$ is the identity homomorphism on $\pi_1(X, a_0)$. Thus r_* is both a left and a right inverse for i_*, and so we conclude that i_* is an isomorphism.

(b) (2 points each) Each of the following topological spaces has a fundamental group isomorphic either to the trivial group, the integers, or the fundamental group of the "figure eight" (shown below). Use part (a) to give the fundamental group of each space. Describe an appropriate deformation retract as precisely as possible, whether with a formula or in words or with pictures or a combination thereof. If you are able to give the deformation retraction, you do not need to prove that it is a deformation retraction.

- The solid torus, $D^2 \times S^1$.

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers.

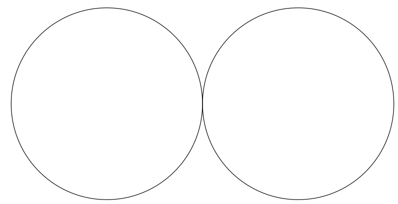

Figure 3: The "figure eight": two circles tangent at a point.

- The torus $S^1 \times S^1$ with a point removed.

  Deformation retracts to an embedding of the Figure-8, hence its fundamental group is that of the Figure-8 (for those of you who have had algebra, this group is the free group on two generators).

- The annulus $S^1 \times I$.

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers.

- The infinite cylinder $S^1 \times \mathbb{R}$.

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers.

- $\mathbb{R}^2$ with the origin removed.

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers.

The following are subsets of $\mathbb{R}^2$:

- The union of the x-axis and the y-axis.

  Deformation retracts to the origin, hence its fundamental group is trivial.

- $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 > 1\}$

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers. Note that one cannot use here the usual embedding of S^1 in $\mathbb{R}^2$; one must instead use a circle of radius > 1.

- $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \geq 1\}$

  Deformation retracts to an embedding of S^1, hence its fundamental group is the integers.

- $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

  Deformation retracts to the origin, hence its fundamental group is trivial.

- The union of S^1 and the x-axis.

  Deformation retracts to the "theta space" of Example 3, p. 362. Looking at Example 2 as well on the same page, we see that both the "theta space" and the Figure-8 are deformation retracts of $\mathbb{R}^2$ with two points removed, and thus they have the same fundamental group. Note that the Figure-8 itself is NOT a retract of theta space, since it does not actually sit inside theta space as a subspace.