MATH 453
SOLUTIONS TO ASSIGNMENT 11
DECEMBER 3, 2004

Exercise 2 from Section 51, page 330
This is a special case of problem 3 below. □

Exercise 3 from Section 51, page 330

(a) The formula $F(x,t) = (1-t)x$ is a homotopy between the identity map on either I or $\mathbb{R}$ to the constant map at 0.

(b) Suppose X is contractible and let $F : X \times I \to X$ be a nullhomotopy with $F(x,0) = x$ and $F(x,1) = x_0$ for all x. If $x, y \in X$, $F(x,t)$ and $F(y,t)$ are paths from x and y to x_0 respectively. Then $F(x,t) * \overline{F(y,t)}$ is a path from x to y.

(c) Y is contractible, so $i_Y \simeq e_{y_0}$ where e_{y_0} is the constant map at y_0. Thus given any map $f : X \to Y$, we have $i_Y \circ f \simeq e_{y_0} \circ f$. Since $i_Y \circ f = f$, we see that any map is homotopic to the constant map that takes all of X to y_0.

(d) Let $i_X \simeq e_{x_0}$ and $[f], [g] \in [X, Y]$. Then $f = f \circ i_X \simeq f \circ e_{x_0} = e_{f(x_0)}$ and similarly $g \simeq e_{g(x_0)}$. Hence we need only show that $e_{f(x_0)} \simeq e_{g(x_0)}$. Since Y is path connected, we have a path $\alpha : I \to Y$ from $f(x_0)$ to $g(x_0)$. This gives the required homotopy $F(x,t) = \alpha(t)$ from $e_{f(x_0)}$ to $e_{g(x_0)}$. □

Exercise 2 from Section 52, page 334
This is a direct computation using the definition of $\hat{\gamma}$: given $[f] \in \pi_1(X, x_0)$, we have

$$\begin{aligned}\hat{\gamma}([f]) &= [\overline{\gamma}] * [f] * [\gamma] = [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta] \\ &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] = \hat{\beta}([\overline{\alpha}] * [f] * [\alpha]) = \hat{\beta}(\hat{\alpha}(f)).\end{aligned}$$

□

Exercise 4 from Section 52, page 335
If $[\alpha] \in \pi_1(A, a_0)$, then $\alpha : (I, \{0,1\}) \to (X, a_0)$ and $r \circ \alpha = \alpha$. Hence $r_*([\alpha]) = [\alpha]$. In words, a loop in A is also a loop in X and r leaves it untouched. □

Exercise 3 from Section 53, page 341
If you are following the text and assuming only that B is connected, proceed as follows: Let B_k be the set of points b in B for which $p^{-1}(b)$ has k points. Then B_k is nonempty, since $b_0 \in B_k$. It is open, for if $b \in B_k$ and U is an evenly covered neighborhood of b, each point in U has k points in its pre-image, i.e. $U \subset B_k$. Exactly the same argument applied to a point not it B_k shows that $B - B_k$ is open as well. Hence B_k is both open and closed and so is all of B.

If we assume that all spaces are path connected, as in class, we can use the path lifting lemma to set up a bijection $f : p^{-1}(a) \to p^{-1}(b)$ for any $a, b \in B$. Suppose α is a path in B from a to b. For each $x \in p^{-1}(a)$, there is a unique lift of α starting at x, say $\tilde{\alpha}$. Define f by $f(x) = \tilde{\alpha}(1)$. Now note that if $\tilde{\alpha}$ is a lift of α, then $\overline{\tilde{\alpha}}$ is a lift of $\overline{\alpha}$ starting at $\tilde{\alpha}(1)$, so applying the path lifting lemma to $\overline{\alpha}$ shows that f is surjective. f is also injective, for if $f(x) = f(y)$ for $x \neq y$, then there would be two lifts of $\overline{\alpha}$ starting at $f(x)$. Hence we see that if $p^{-1}(b_0)$ has k elements for some b_0, then so does $p^{-1}(b)$ for every $b \in B$.

$\square$