Math 453
Solutions to Assignment 9
November 11, 2004

Exercise 6 from Section 28, page 181
X is a metric space, so it is Hausdorff. Thus X is compact Hausdorff and we need only check that f is a continuous bijection. Continuity and injectivity is immediate since f is an isometry.

Suppose that f is not onto, and let $a \in X - f(X)$. $f(X)$ is compact, being the continuous image of a compact set, so it is closed. Thus $X - f(X)$ is open and we can find an $\varepsilon > 0$ such that $B(a, \varepsilon)$ is disjoint from $f(X)$. Set $x_0 = a$ and for $n > 0$, $x_n = f(x_{n-1}) = f^n(a)$. If $n > m$, we have $d(x_n, x_m) = d(f^n(a), f^m(a)) = d(f^{n-m}(a), a) > \varepsilon$ since $f^{n-m}(a)$ is in $f(X)$ and a isn't. Thus $\{x_n\}$ has no convergent subsequence and this contradicts the compactness of X. □

Exercise 7 from Section 28, page 181

(a) First note that since X is compact, $\{d(x, y) \mid x, y \in X\}$ is a compact subset of $\mathbb{R}$. Thus $\operatorname{diam} X$, defined to be the supremum of the above set, is finite.

Let $A_0 = X$, and $A_n = f(A_{n-1})$ for $n > 0$. It is clear that each A_n is compact (hence closed) and $A_{n+1} \subset A_n$, so the finite intersection property implies that their intersection (call it A) is non-empty. Note that A contains all the fixed points of f.

Furthermore, $\operatorname{diam} A \leq \operatorname{diam} A_n \leq \alpha^n \operatorname{diam} X$ for each n. Hence $\operatorname{diam} A = 0$ and A is a singleton, say $\{a\}$. Now clearly $f(A) \subset A$, so $f(a) = a$ and we have found the unique fixed point.

(b) As before, let $A_0 = X$, $A_n = f(A_{n-1})$ and $A = \bigcap A_n$. We claim that $A = f(A)$. The inclusion $f(A) \subset A$ is clear; for the other inclusion, suppose $x \in A$. We will find an a such that $f(a) = x$. Since $x \in A$, for each n, $x \in A_{n+1}$, so there is an x_n such that $f^{n+1}(x_n) = x$. Let $y_n = f^n(x_n)$ and a be the limit of a convergent subsequence of $\{y_n\}$ (which exists since X is compact).

Now $f(y_n) = f(f^n(x_n)) = f^{n+1}(x_n) = x$ for all n, so in particular this holds for the above subsequence. Hence $f(a) = x$. Furthermore, each closed A_n contains all but a finite number of the y_n's, so contains the limit a of a subsequence. Thus $a \in A$.

This implies that $A = f(A)$. However, since f is a shrinking map and A is compact, $\operatorname{diam} A < \operatorname{diam} f(A)$ if A has more than one point. Thus we conclude that A has only one point, the required fixed point of f.

(c) A direct computation yields

$$|f(x) - f(y)| = |x - y| \left|1 - \tfrac{1}{2}(x + y)\right|,$$

and the result is immediate.

(d) In this case we have

$$f(x) - f(y) = \frac{1}{2}(x-y)\Big(1 + \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}}\Big),$$

so

$$|f(x) - f(y)| \leq \frac{1}{2}|x-y|\Big(1 + \frac{|x|+|y|}{\sqrt{x^2+1} + \sqrt{y^2+1}}\Big).$$

Hence f is a shrinking map that is not a contraction. $f(x) = x$ implies the nonsensical relation $x = \sqrt{x^2+1}$ so it has no fixed point. □

Exercise 1 from Section 29, page 186

We will show that there is no point at which $\mathbb{Q}$ is locally compact. Suppose not, and assume that $\mathbb{Q}$ is locally compact at q. This means that there is a compact set C containing a neighborhood $U = (a, b) \cap \mathbb{Q}$ of q. Let s be an irrational number in (a, b) and $\{x_n\}$ a sequence in U that converges to s. Then $\{x_n\}$ is a sequence in C that has no convergent subsequence in C, contradicting the compactness of C. □

Exercise 5 from Section 29, page 186

Let $Y_1 = X_1 \cup \{\infty_1\}$ and $Y_2 = X_2 \cup \{\infty_2\}$ be the one-point compactifications of X_1 and X_2 respectively. We extend f to $\tilde{f} : Y_1 \to Y_2$ in the obvious way: $\tilde{f}(\infty_1) = \infty_2$ and $\tilde{f}|_{X_1} = f$. Since Y_1 is compact and Y_2 is Hausdorff, we need only check that $\tilde{f}$ is a continuous bijection. The second condition is immediate, so we show that $\tilde{f}$ is continuous. Suppose U is open in Y_2. If U does not contain ∞_2, it is open in X_2. Hence $\tilde{f}^{-1}(U) = f^{-1}(U)$ is open in Y_1. If $U = Y_2 - C$ for a compact subset C of X_2, $\tilde{f}^{-1}(U) = Y_1 - f^{-1}(C)$. Since f is a homeomorphism, $f^{-1}(C)$ is compact in X_1 and so $\tilde{f}^{-1}(U)$ is open in Y_1. □

Exercise 10 from Section 29, page 186

X is locally compact at x, so there is a compact set C containing a neighborhood A of x. Then $U \cap A$ is a neighborhood of x contained in the compact Hausdorff space C. The subspace $C - (A \cap U)$ is closed, so it is compact, and thus, by Lemma 26.4, we can find disjoint open sets V and W containing x and $C - (U \cap A)$, respectively. V is the required neighborhood of x: $\overline{V}$ is a closed subset of C so is compact, and $\overline{V} \subset U$ since $\overline{V} \subset C - W \subset U \cap A$. □