

MATH 453  
SOLUTIONS TO ASSIGNMENT 8  
NOVEMBER 6, 2004

*Exercise 2 from Section 26, page 171*

- (a) Let  $X$  be a subspace of  $\mathbb{R}$  in the finite complement topology, and let  $\{U_\alpha\}$  be an open cover for  $X$ . Pick a particular  $U_{\alpha_0} = \mathbb{R} - A_{\alpha_0}$ , where  $A_{\alpha_0}$  is finite. Then  $X - U_{\alpha_0} \subset A_{\alpha_0}$  is finite, say  $X - U_{\alpha_0} = \{a_1, a_2, \dots, a_n\}$ . For each  $a_i$ , we can find an  $U_{\alpha_i}$  that contains it, so  $\{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_n}\}$  is a finite subcover.
- (b)  $[0, 1]$  is not compact in this “countable complement” topology. For each  $n \in \mathbb{Z}^+$ , let  $U_n = [0, 1] - \{\frac{1}{i} \mid i > n\}$ . Then  $\{U_n\}_{n \in \mathbb{Z}^+}$  is an open cover for  $[0, 1]$  that has no finite subcover.  $\square$

*Exercise 3 from Section 26, page 171*

Let  $A_1, \dots, A_n$  be compact subspaces of  $X$ ; let  $A = \bigcup_{i=1}^n A_i$ . Suppose  $\mathcal{C}$  is an open cover for  $A$ . Then  $\mathcal{C}$  is also an open cover for each  $A_i$ , so we can find finite subcovers  $\mathcal{C}_i$ . Their union is the required finite subcover for  $A$ . Hence  $A$  is compact.  $\square$

*Exercise 5 from Section 26, page 171*

By Lemma 26.4, for each  $a \in A$ , we can find disjoint open sets  $U_a$  and  $V_a$  such that  $a \in U_a$  and  $B \subset V_a$ . The collection  $\{U_a \mid a \in A\}$  is an open cover for  $A$ , so by compactness, we can find a finite subcover  $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\}$ . Let  $U = \bigcup_{i=1}^n U_{a_i}$  and  $V = \bigcap_{i=1}^n V_{a_i}$ . Then  $U$  and  $V$  are disjoint, and  $A \subset U$  and  $B \subset V$ .  $\square$

*Exercise 11 from Section 26, page 171*

Suppose that  $Y$  is not connected, so there is a separation  $C \cup D$ . Then  $C$  and  $D$  are both closed in  $Y$  and hence in  $X$  since  $Y$  is closed in  $X$ . Therefore  $C$  and  $D$  are compact and by problem 26.5 we can find disjoint open sets  $U$  and  $V$  containing  $C$  and  $D$  respectively.

Now the collection  $\{A - (U \cup V) \mid A \in \mathcal{A}\}$  is a collection of closed sets ordered by proper inclusion. Furthermore, each  $A - (U \cup V)$  is non-empty lest  $A \cap U$  and  $A \cap V$  be a separation. Hence the collection has the finite intersection property and so has a non-empty intersection. This, however, is not possible since their intersection is simply  $Y - (U \cup V)$ . Thus  $Y$  cannot have a separation.  $\square$

*Exercise 3 from Section 27, page 177*

- (a) Recall that the  $K$ -topology is generated by the basis  $\{(a, b), (a, b) - K \mid a < b\}$ , where  $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ . For each  $i$ , let  $U_i = (\frac{1}{i}, 2) \cup (-1, 1) - K$ ; the open cover  $\{U_i\}$  of  $[0, 1]$  does not have a finite subcover, so  $[0, 1]$  is not compact.
- (b)  $(-\infty, 0)$  clearly has the usual topology as a subspace of  $\mathbb{R}_K$ . For  $(0, \infty)$ , a basis for the subspace topology is given by  $\{(a, b), (a, b) - K \mid 0 \leq a < b\}$ . Since  $(a, b) - K$  is

a union of open intervals if  $a \geq 0$ , this basis generates the usual topology on  $(0, \infty)$ . Thus both  $(-\infty, 0)$  and  $(0, \infty)$  are connected. Furthermore, 0 is in the closure of both sets, since any basic neighborhood of 0 must intersect both. Hence, by Theorem 23.4,  $(-\infty, 0]$  and  $[0, \infty)$  are connected. Finally, we have  $\mathbb{R}_K = (-\infty, 0] \cup [0, \infty)$ , so  $\mathbb{R}_K$  is connected.

- (c) To avoid confusion, I'll use a subscript "K" to indicate that an interval is to be regarded as a subspace of  $\mathbb{R}_K$ , so  $[a, b]$  has the usual topology, while  $[a, b]_K$  has the subspace topology inherited from  $\mathbb{R}_K$ .

Let  $p, q \in \mathbb{R}_K$  be such that  $p < 0$  and  $q > 1$ . If  $\mathbb{R}_K$  were path connected, then there is a continuous function  $f : [0, 1] \rightarrow \mathbb{R}_K$  such that  $f(0) = p$  and  $f(1) = q$ . Since the  $K$ -topology is finer than the usual topology on  $\mathbb{R}$ ,  $f$ , when regarded as a map to  $\mathbb{R}$ , is also continuous. By the intermediate and extreme value theorems,  $f([0, 1])$  is an interval that contains  $p$  and  $q$ .

Going back to  $\mathbb{R}_K$ ,  $f([0, 1])$  is compact, since  $f$  is continuous and  $[0, 1]$  is compact. Moreover,  $[0, 1]_K$  is a closed subspace of  $f([0, 1])$ , so this implies that  $[0, 1]_K$  is compact. However, in (a) we showed that  $[0, 1]_K$  is not compact, so there cannot be a path from  $p$  to  $q$  and  $\mathbb{R}_K$  cannot be path connected.  $\square$

*Exercise 5 from Section 27, page 178*

Let  $A$  denote the union  $\bigcup A_n$ .  $A$  has empty interior if and only if given any non-empty open set  $U_0$ ,  $U_0 \not\subset A$ , or, in other words, there is an element  $x \in U_0$  that is not in  $A$ . We'll use a method similar to the proof of Theorem 27.7 to produce such an  $x$ .

Consider the set  $A_1$ . It cannot contain  $U_0$  since it has empty interior; pick an element  $y \in U_0$  that is not in  $A_1$ . Being a closed subspace of the compact space  $X$ ,  $A_1$  is compact. Hence Lemma 26.4 implies that there are disjoint open sets  $V, W$  that contain  $y$  and  $A_1$ , respectively. Now  $B = X - (U_0 \cap V)$  is closed and  $y \notin B$ , so, by Lemma 26.4 again, we can find disjoint open sets  $U_1$  and  $W'$  that contain  $y$  and  $B$ , respectively. Note that  $\bar{U}_1$  is disjoint from  $W'$  (since  $X - W'$  is closed), so  $\bar{U}_1 \subset U_0 \cap V$ . Thus  $\bar{U}_1$  is contained in  $U_0$  and is disjoint from  $A_1$ . In summary, given the non-empty open set  $U_0$  and the closed set  $A_1$  with empty interior, we can find an open set  $U_1$  such that

$$\begin{aligned}\bar{U}_1 \cap A_1 &= \emptyset, \\ \bar{U}_1 &\subset U_0.\end{aligned}$$

In general, given the non-empty open set  $U_{n-1}$  and the closed set  $A_n$ , we can choose an open set  $U_n$  such that

$$\begin{aligned}\bar{U}_n \cap A_n &= \emptyset, \\ \bar{U}_n &\subset U_{n-1}.\end{aligned}$$

This gives us a nested sequence  $\bar{U}_1 \supset \bar{U}_2 \supset \dots$  of non-empty subsets of  $X$ . The collection  $\{\bar{U}_n\}$  thus has the finite intersection property, so, by the compactness of  $X$ , the

intersection  $\bigcap \bar{U}_n$  must be non-empty. Any  $x \in \bigcap \bar{U}_n$  will be in  $U_0$  (since  $\bar{U}_1 \subset U_0$ ) and not in  $A$  (since  $\bar{U}_n$  is disjoint from  $A_n$ ). Thus  $A$  has empty interior.  $\square$

*Exercise 6 from Section 27, page 178*

- (a) We will show that any two points of  $C$  lie in different components; it follows that only singletons are connected. Suppose that  $a < b$  are in  $C$ . The interval  $(a, b)$  contains a closed interval of the form  $[k/3^n, (k+1)/3^n]$  for some sufficiently large  $n$ . Hence either the corresponding open interval has been removed by stage  $n$  or its middle third will be removed at stage  $n+1$ . This means that  $[0, (2k+1)/2 \cdot 3^n] \cap C$  and  $[(2k+1)/2 \cdot 3^n, 1] \cap C$  form a separation of  $C$  with  $a$  in the first set and  $b$  in the second. Thus  $a$  and  $b$  are not in the same component.
- (b)  $C$  is an intersection of closed sets  $A_n$  and so is closed. Being a closed subspace of a compact space makes it compact.
- (c) The first claim is immediate via induction. The endpoints are in  $C$  since they are never removed from any of the  $A_n$ 's.
- (d) Suppose  $x \in C$  is an isolated point, so  $\{x\}$  is an open set. This means that it is the intersection of an open set  $U$  in  $[0, 1]$  with  $C$ . If  $x$  is 0 or 1,  $U$  contains an interval of the form  $[0, a)$  or  $(a, 1]$  respectively; otherwise it contains one of the form  $(a, b)$ . Letting  $I$  denote any of the above intervals, we see that  $C \cap I$  is a subset of  $\{x\}$ . This is not possible, since any such interval must contain an infinite number of points of  $C$ : it contains infinitely many of the endpoints in (c).
- (e) This is immediate from the previous parts and Theorem 27.7.  $\square$