Math 453
Solutions to Assignment 7
November 1, 2004

Exercise 3 from Section 23, page 152
Let $B_\alpha = A \cup A_\alpha$. Then B_α is connected for each α since A and A_α are connect and have a point in common. Now $\bigcup B_\alpha = A \cup \bigcup A_\alpha$ and the B_α's have a point in common (A is nonempty) so $A \cup \bigcup A_\alpha$ is connected. $\square$

Exercise 10 from Section 24, page 158
Suppose $x_0 \in U$ and let V be the set of points in U that can be joined to x_0 by a path in U. V is non-empty since it contains x_0. If $x \in V$, then there is an open ball $B(x, \varepsilon)$ around x contained in U as U is open. Each point of $B(x, \varepsilon)$ can be connected to x via a straight line (path) in $B(x, \varepsilon)$, and x can be connected to x_0 via a path in U. Hence each element of $B(x, \varepsilon)$ can be joined to x_0 via a path in U. Thus $B(x, \varepsilon) \subset V$ and so V is open.

Now let x be a limit point of V in U. Again we can find an open ball $B(x, \varepsilon)$ around x that is contained in U. Since x is a limit point of V, $B(x, \varepsilon) \cap V$ is nonempty; let y be a point in the intersection. Then there is a path from x to y and a path from y to x_0. All these paths are in U, so concatenation produces a path from x to x_0 in U. Hence $x \in V$ and V is closed.

Putting all these together we see that V is a nonempty, closed and open subset of the connected set U and so must equal U. Since V is clearly path connected, so is U. $\square$

Exercise 7 from Section 25, page 162
If we can show that every connected neighborhood of p contains all the points a_i, then X cannot be locally connected at p since not every neighborhood of p need contain all the a_i's. Suppose U is a connected neighborhood of p. Then U contains all but finitely many a_i. Furthermore, if $a_n \in U$, then U intersects some of the "spikes" that emanate from a_{n-1}. In order for U to be connected, we must have $a_{n-1} \in U$. Thus U contains each a_i.

We now show that X is weakly locally connected at p. This means given any neighborhood U of p, there is a connected subspace Y of X contained in U that contains a neighborhood of p. To aid the description of Y, consider X to be a subspace of $\mathbb{R}^2$ with $p = (0, 0)$ and each a_i on the positive x-axis. Then $U = X \cap V$ for some open set V in $\mathbb{R}^2$. V is open, contains p, and $a_i \to p$, so it contains some open ball $B(p, a_n)$. I am being lazy here and using a_n to denote both the point and its (positive) x-coordinate. Set $Y = X \cap B(p, a_{n+1}) \cup \{a_{n+1}\}$. Then $Y \subset U$ and contains the neighborhood $X \cap B(p, a_{n+1})$ of p. Hence X is weakly locally connected at p.

$\square$