MATH 453
SOLUTIONS TO ASSIGNMENT 6
OCTOBER 16, 2004

Exercise 4 from Section 20, page 127

(a) Since each of their component functions are continuous, all three of f, g and h are continuous when $\mathbb{R}^\omega$ is given the product topology.

On the other hand, if $\mathbb{R}^\omega$ is given the box topology, none of them is continuous. To see this, look at the open set $U = \prod_{n=1}^{\infty}(-1/n^2, 1/n^2)$ in $\mathbb{R}^\omega$. The inverse image of U under each of f, g and h is $\{0\}$ and hence is not open.

Things are a little more interesting in the uniform topology: f is not continuous, since $f^{-1}(V) = \{0\}$ if V is the product of countably many copies of $(-1, 1)$. Let k denote either g or h. Then $\bar{\rho}(k(s) - k(t)) = \bar{d}(s - t)$, so if we have an ε-ball $B_{\bar{\rho}}(f(s), \varepsilon)$ about $f(s)$ in the $\bar{\rho}$ metric, the ε-ball around s (i.e. the interval $(s - \varepsilon, s + \varepsilon)$) in $\mathbb{R}$ will be mapped into $B_{\bar{\rho}}(f(s), \varepsilon)$. Hence both g and h are continuous in the uniform topology.

(b) First observe that if $(\mathbf{a}_n)$ denotes any of the given sequences, then $\pi_i(\mathbf{a}_n) \to 0$ for each i. It follows from Exercise 18.6 (and a similar proof for the uniform topology) that the only point the sequences can possibly converge to is $\mathbf{0} = (0, 0, \dots)$. Furthermore, we can also conclude that $\mathbf{a}_n \to \mathbf{0}$ in the product topology.

Now consider the uniform topology. $(\mathbf{w}_n)$ does not converge to $\mathbf{0}$ (and hence does not converge) since the neighborhood V (from part (a) above) does not contain any of the $\mathbf{w}_n$'s. The other three sequences, which we denote by $(\mathbf{b}_n)$, do converge to $\mathbf{0}$: given any ε-ball B around $\mathbf{0}$, choosing $N > 1/\varepsilon$ will ensure that $\mathbf{b}_n \in B$ whenever $n \geq N$.

Finally for the box topology. This is finer than the uniform topology, so $(\mathbf{w}_n)$ does not converge either. In fact, this topology is so fine that even $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ do not converge. To see this, note that the neighborhood U from (a) does not contain any $\mathbf{x}_n$ or $\mathbf{y}_n$. $(\mathbf{z}_n)$, however, do converge to $\mathbf{0}$, since given a basic neighborhood $W = \prod_{n=1}^{\infty} W_n$ of $\mathbf{0}$, we need only choose N so that $n \geq N$ ensures $\pi_1(\mathbf{z}_n) \in W_1$ *and* $\pi_2(\mathbf{z}_n) \in W_2$ to have $\mathbf{w}_n \in W$ for $n \geq N$. Such a choice of N is possible since both $\pi_1(\mathbf{w}_n)$ and $\pi_2(\mathbf{w}_n)$ converge to 0.

□

Exercise 5 from Section 20, page 127

The closure of $\mathbb{R}^\infty$ consists of all sequences that converges to 0. Let $x = (x_1, x_2, \dots)$ be a sequence that converges to 0, and U a neighborhood of x. Then U contains a basic neighborhood $V = \prod_{n=1}^{\infty}(x_n - \varepsilon, x_n + \varepsilon)$ where $\varepsilon > 0$. Since $x_n \to 0$, we can find an N

such that $|x_n| < \varepsilon$ whenever $n > N$. Thus $(x_1, x_2, \ldots, x_N, 0, 0, \ldots)$ is an element in V that is also in $\mathbb{R}^\infty$. To see that no other elements of $\mathbb{R}^\omega$ can be in the closure of $\mathbb{R}^\infty$, suppose that $y = (y_1, y_2, \ldots)$ is a sequence that does not converge to 0. By definition, this means there is an $\varepsilon > 0$ such that for every N, there is an $n > N$ satisfying $|y_n| \geq \varepsilon$. Then $\prod_{n=1}^\infty (y_n - \varepsilon, y_n + \varepsilon)$ is a neighborhood of y disjoint from $\mathbb{R}^\infty$. □

Exercise 2 from Section 21, page 133

We need to show that $f : X \to f(X)$ is a homeomorphism (I am being sloppy with notation here, but since we shall not be referring to $f : X \to Y$, no confusion can arise). If $f(x_1) = f(x_2)$, then

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0,$$

so $x_1 = x_2$. Hence f is a bijection, surjectivity being a given. Thus f^{-1} is defined, and furthermore, it is also isometric, that is,

$$d_Y(y_1, y_2) = d_X(f^{-1}(y_1), f^{-1}(y_2)).$$

Now, any isometric map is continuous, since we need only take $\delta = \varepsilon$ in Theorem 21.1. Hence both f and f^{-1} are continuous and f is an isometric imbedding. □

Exercise 4 from Section 21, page 134

For any $x \in \mathbb{R}_\ell$, $U_n = [x, x + 1/n)$ is a countable basis at x. This is a basis at x because any neighborhood U of x contains a basic open set $[x, y)$ (for $y > x$), which in turn contains some U_n.

For $x \times y \in I \times I$ in the dictionary order topology we can use essentially the same argument. However, there are a few cases to consider:

(a) 0×0:
$U_n = [0 \times 0, 0 \times \frac{1}{n})$.

(b) 1×1:
$U_n = (1 \times 1 - \frac{1}{n}, 1 \times 1]$.

(c) $x \times y$ where $0 < y < 1$:
$U_n = (x \times y - \frac{1}{N+n}, x \times y + \frac{1}{N+n})$ for some N.

(d) $x \times 0$:
$U_n = (x - \frac{1}{N+n} \times 0, x \times \frac{1}{n})$ for some N.

(e) $x \times 1$:
$U_n = (x \times 1 - \frac{1}{n}, x + \frac{1}{N+n} \times 1)$ for some N.

In cases (c-e) N is merely a technicality needed to ensure that we stay within the square $I \times I$. □

Exercise 2 from Section 22, page 144

(a) The existence of f implies that p is surjective, since the identity on Y is. Hence we only need to check that if $p^{-1}(U)$ is open then U is open. Since $p \circ f$ is the identity, $U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U))$. f is continuous and $p^{-1}(U)$ is assumed to be open, so the result follows.

(b) Let $i : A \to X$ be the inclusion, i.e. $i(a) = a$ for all $a \in A$. Then $r \circ i$ is the identity on A so the result follows from (a).

□

Exercise 4 from Section 22, page 145

(a) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = x + y^2$. Then g is surjective and there is an $f : X^* \to \mathbb{R}$ such that $g = f \circ q$. Using Corollary 22.3, we need to check that g is a quotient map in order to claim that X^* is homeomorphic to $\mathbb{R}$. If we identify $\mathbb{R}$ with the x-axis, we have $U = g^{-1}(U) \cap \mathbb{R}$ for any $U \subset \mathbb{R}$, so U is open if and only if $g^{-1}(U)$ is open. Thus g is a quotient map and we are done.

(b) Do the same thing as (a) with $g : \mathbb{R}^2 \to [0, \infty)$ given by $g(x, y) = x^2 + y^2$.

□