MATH 453
SOLUTIONS TO ASSIGNMENT 5
OCTOBER 8, 2004

Exercise 4 from Section 18, page 111

We check that f is an imbedding; the argument for g is similar. f is clearly a bijection between X and $f(X) = X \times y_0$, so we need only show that it and its inverse are continuous. It follows from the definition of the subspace topology that the open sets of $X \times y_0$ are of the form $U \times y_0$ where U is open in X. Since $f(U) = U \times y_0$, it is immediate that both f and f^{-1} are continuous. $\square$

Exercise 5 from Section 18, page 111

The map $f : [0,1] \to [a,b]$ given by $f(x) = a + (b-a)x$ is clearly a homeomorphism that restricts to a homeomorphism from $(0,1)$ to (a,b). $\square$

Exercise 6 from Section 18, page 111

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if x is rational and $f(x) = x$ otherwise. We will show that f is continuous only at 0 by using the sequence criterion for continuity. If $\{a_i\}$ is any sequence that converges to 0, then $\{f(a_i)\}$ is obtained from $\{a_i\}$ by replacing the rational a_i's by 0, so it also converges to 0. Hence f is continuous at 0. Now suppose $x \neq 0$. Pick a sequence of rational numbers $\{a_i\}$ and a sequence of irrational numbers $\{b_i\}$ that both converge to x. Then $f(a_i) = 0$ for each i, so $\{f(a_i)\}$ converges to 0, while $f(b_i) = b_i$, so $\{f(b_i)\}$ converges to x. Hence f cannot be continuous at x, whether x is rational or not. $\square$

Exercise 12 from Section 18, page 112

(a) F is symmetric in x and y, so it suffices to check that it is continuous in x with y fixed. If $y_0 \neq 0$, $F(x \times y_0) = xy_0/(x^2 + y_0^2)$ is clearly continuous. $F(x \times 0)$ is identically zero and so is continuous too.

(b) $g(x) = 1/2$ if $x \neq 0$ and $g(0) = 0$ so g is not continuous at 0.

(c) Define a function $\Delta : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $\Delta(x) = x \times x$. Δ is continuous since each component is. If F were continuous, then $g = F \circ \Delta$ would be continuous, contradicting part (b).

$\square$

Exercise 6 from Section 19, page 118

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x}$ in the product topology. For a fixed α, let U_α be a neighborhood of $\pi_\alpha(\mathbf{x})$. We need to check that U_α contains $\pi_\alpha(\mathbf{x}_i)$ for i larger than some N. This follows from the fact that $U = \prod_\beta V_\beta$, where $V_\beta = X_\beta$ for $\beta \neq \alpha$ and $V_\beta = U_\alpha$ is a neighborhood of $\mathbf{x}$ and so contains $\mathbf{x}_i$ for i larger than some N.

Conversely suppose that $\{\pi_\alpha(\mathbf{x}_i)\}$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Let U be a basic neighborhood of $\mathbf{x}$, so that $U = \prod_\alpha U_\alpha$ and $U_\alpha = X_\alpha$ except when $\alpha = \alpha_k$, $k = 1, 2, \ldots, K$. Now for each α_k, $\{\pi_{\alpha_k}(\mathbf{x}_i)\}$ converges to $\pi_{\alpha_k}(\mathbf{x})$, so there is an N_k such that $\pi_{\alpha_k}(\mathbf{x}_i) \in U_{\alpha_k}$ whenever $i > N_k$. Set $N = \max\{N_k \mid k = 1, 2, \ldots, K\}$. Then for all $i > N$, we have $\mathbf{x}_i \in U$ and so $\{\mathbf{x}_i\}$ converges to $\mathbf{x}$.

This is not true in the box topology. Take the space to be $\mathbb{R}^\omega$ and let $\mathbf{x}_i$ be the sequence consisting of all zeros except a 1 at the ith position. Then each $\{\pi_\alpha(\mathbf{x}_i)\}$ converges to 0 but the neighborhood $\prod_n(-1/n, 1/n)$ of $(0, 0, \ldots)$ contains none of the $\mathbf{x}_i$'s. □

Exercise 8 from Section 19, page 118

h is a homeomorphism in both topologies. It is clear that h is a bijection with inverse of the same form. Hence it suffices to check that h is continuous. We can choose bases for both the box and product topologies to be

$$\mathscr{B} = \{\Pi_{n \in \mathbb{Z}^+} U_n \mid U_n = \mathbb{R} \text{ or } U = (a, b)\}\,.$$

There is an extra condition that $U_n = \mathbb{R}$ for all but finitely many n for the product topology, but that is not important here. To show that h is continuous, it suffices to check that $h^{-1}(B)$ is open for $B \in \mathscr{B}$. If f is any of the component functions of h, then f^{-1} takes intervals (or $\mathbb{R}$) to intervals (or $\mathbb{R}$), so $h^{-1}(B)$ is actually in $\mathscr{B}$. Hence h is continuous. □