MATH 453
SOLUTIONS TO ASSIGNMENT 4
OCTOBER 7, 2004

Exercise 2 from Section 17, page 100

Using Theorem 17.2, A is closed in Y implies that it is the intersection of Y and a closed subset C of X: $A = C \cap Y$. Since Y is given to be closed in X, A is the intersection of two closed sets in X, and hence is closed in X. □

Exercise 6 from Section 17, page 101

(a) $\bar{B}$ is a closed set that contains B, so it contains A as well. Hence $\bar{A} \subset \bar{B}$.

(b) $\bar{A} \cup \bar{B}$ is closed, being the finite union of closed sets, and contains $A \cup B$, so it must contain the closure $\overline{A \cup B}$ as well. The other inclusion is a special case of (c).

(c) For each α, $A_\alpha \subset \overline{\bigcup A_\alpha}$, so $\bar{A}_\alpha \subset \overline{\bigcup A_\alpha}$. Consequently, $\bigcup \bar{A}_\alpha \subset \overline{\bigcup A_\alpha}$. An example where equality fails is given by $A_n = [1/n, 1]$, $n \in \mathbb{Z}_+$ in $\mathbb{R}$. Each A_n is closed, so $\bigcup \bar{A}_n = \bigcup A_n = (0, 1]$. However, $\overline{\bigcup A_n} = \overline{(0, 1]} = [0, 1]$.

□

Exercise 7 from Section 17, page 101

The given "proof" fails because, while every neighborhood U of x must intersect some A_α, each such neighborhood may intersect a different A_α, leaving x in the closure of none. For an example that illustrates this, consider the collection $A_n = \{1/n\}$, $n \in \mathbb{Z}_+$ of subsets of $\mathbb{R}$. Their union has the single limit point 0, and every neighborhood of 0 intersects some A_n, but not always the same A_n. □

Exercise 8 from Section 17, page 101

(a) As in 6(b) above, $\bar{A} \cap \bar{B}$ is a closed set containing $A \cap B$, so must contain its closure $\overline{A \cap B}$. The other inclusion fails though: take $A = (0, 1)$ and $B = (-1, 0)$. $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$, while $\bar{A} \cap \bar{B} = \{0\}$.

(b) The inclusion $\overline{\bigcap A_\alpha} \subset \bigcap \bar{A}_\alpha$ follows via an obvious generalization of part (a), while the other inclusion fails since it already failed for the special case of two sets.

(c) Again, we don't have equality: take $A = [0, 1]$ and $B = (0, 1)$, so $\overline{A - B} = \overline{\{0, 1\}} = \{0, 1\}$ while $\bar{A} - \bar{B} = \emptyset$. We do, however, have $\bar{A} - \bar{B} \subset \overline{A - B}$. First note that $A = (A - B) \cup (A \cap B)$, so, by 6(b) above, $\bar{A} = \overline{A - B} \cup \overline{A \cap B}$. Similarly, $\bar{B} = \overline{B - A} \cup \overline{A \cap B}$. Hence $\overline{A \cap B}$ is a subset of both $\bar{A}$ and $\bar{B}$, and so $\bar{A} - \bar{B} = (\overline{A - B} \cup \overline{A \cap B}) - \bar{B} = \overline{A - B} - \bar{B}$. The result follows immediately.

□

Exercise 9 from Section 17, page 101

$\bar{A} \times \bar{B}$ is a closed set (since its complement is the open set $[(X - \bar{A}) \times Y] \cup [X \times (Y - \bar{B})]$) that contains $A \times B$, so $\overline{A \times B} \subset \bar{A} \times \bar{B}$. For the reverse inclusion, suppose $x \times y \in \bar{A} \times \bar{B}$. Then $x \in \bar{A}$ and $y \in \bar{B}$, so if $U \times V$ is a basic open set (using the usual basis for $X \times Y$) that contains $x \times y$, then U intersects A and V intersects B. Thus $U \times V$ intersects $A \times B$ and we have $x \times y \in \overline{A \times B}$. □

Exercise 13 from Section 17, page 101

Suppose X is Hausdorff. We will show that Δ is closed, or, rather, that $X \times X - \Delta$ is open. If $x \times y \in X \times X - \Delta$, then $x \neq y$, so there exists disjoint neighborhoods U and V of x and y respectively. Thus $U \times V$ is a neighborhood of $x \times y$ that does not intersect Δ, that is, $U \times V \subset X \times X - \Delta$. Hence $X \times X - \Delta$ is open.

Conversely, suppose Δ is closed, so $X \times X - \Delta$ is open. Given $x \neq y$, $x \times y \in X \times X - \Delta$, so there is a basic open set $U \times V$ such that $x \times y \in U \times V \subset X \times X - \Delta$. U and V are the required disjoint neighborhoods of x and y respectively, so X is Hausdorff. □

Exercise 19 from Section 17, page 102

(a) If $x \in \operatorname{Int} A$, then $\operatorname{Int} A$ is a neighborhood of x that does not intersect $X - A$, so $x \notin \overline{X - A}$. Therefore $\operatorname{Int} A \cap \operatorname{Bd} A = \varnothing$.

$\operatorname{Bd} A \subset \bar{A}$ by definition, and $\operatorname{Int} A \subset A \subset \bar{A}$, so $\operatorname{Int} A \cup \operatorname{Bd} A \subset \bar{A}$. For the other inclusion, suppose $x \in \bar{A}$. If some neighborhood U of x does not intersect $X - A$, then $x \in U \subset A$, so $x \in \operatorname{Int} A$; otherwise every neighborhood of x intersects $X - A$, so $x \in \overline{X - A}$, and consequently $x \in \operatorname{Bd} A$. Thus $x \in \bar{A}$ implies $x \in \operatorname{Int} A \cup \operatorname{Bd} A$, or $\bar{A} \subset \operatorname{Int} A \cup \operatorname{Bd} A$.

(b) This follows directly from (a) since A is both open and closed if and only if $\bar{A} = A = \operatorname{Int} A$.

(c) If U is open, $\operatorname{Int} U = U$, and (a) gives $\bar{U} = U \cup \operatorname{Bd} U$. Since this is a disjoint union, we have $\operatorname{Bd} U = \bar{U} - U$. Conversely, suppose $\operatorname{Bd} U = \bar{U} - U$. Then $\bar{U} = \operatorname{Bd} U \cup U$, and the union is disjoint. Thus we have $\operatorname{Bd} U \cup \operatorname{Int} U = \operatorname{Bd} U \cup U$ with both unions being disjoint, so $U = \operatorname{Int} U$ and U is open.

(d) Tempting as this result might be, it is not true. Consider $X = [0, 1]$ as a subspace of $\mathbb{R}$ and $U = (0, 1)$. Then U is open in X, and $\bar{U} = X$, so $\operatorname{Int} \bar{U} = X \neq U$. It is, however, true that $U \subset \operatorname{Int} \bar{U}$ since U is an open subset of $\bar{U}$.

□

Exercise 2 from Section 18, page 111

There is no reason why $f(x)$ should be a limit point of $f(A)$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is a constant function, and $A = \mathbb{R}$, then $f(A)$ has no limit point while every point is a limit point of A. □