MATH 453
SOLUTIONS TO ASSIGNMENT 3
SEPTEMBER 26, 2004

Exercise 1 from Sections 14-16, page 91

First consider A as a subspace of Y. In this topology, a subset B of A is open if and only if it is of the form $B = A \cap V$, where V is an open set in Y. Now Y is a subspace of X, so V is open precisely when it is of the form $V = Y \cap U$ for some U open in X. Putting these together, we get $B = A \cap (Y \cap U) = A \cap U$, since $A \subset Y$. Hence we reach the conclusion that B is open if and only if $B = A \cap U$ for some open set U of X, which is the very definition for B to be open in the subspace topology A inherits from X. $\square$

Exercise 4 from Sections 14-16, page 92

We'll check that $\pi_1 : X \times Y \to X$ is an open map; the argument for π_2 is exactly the same. Let W be an open set in $X \times Y$. Using the definition of the product topology, we can write W as a union of basis elements:

$$W = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha,$$

where J is some indexing set, U_α is open in X, and V_α is open in Y. Unions behave well under a function, namely $f(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha)$, and $\pi_1(U_\alpha \times V_\alpha) = U_\alpha$, so

$$\pi_1(W) = \pi_1\left(\bigcup_{\alpha \in J} U_\alpha \times V_\alpha\right) = \bigcup_{\alpha \in J} \pi_1(U_\alpha \times V_\alpha) = \bigcup_{\alpha \in J} U_\alpha.$$

Since each U_α is open in X, their union, $\pi_1(W)$, is too. Hence π_1 is an open map. $\square$

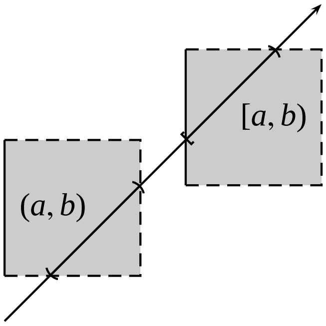
Exercise 6 from Sections 14-16, page 92

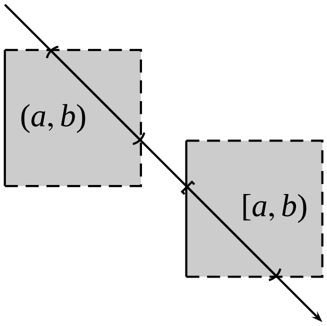
This immediate from Theorem 15.1 and Exercise 8(a) from the previous homework. $\square$

Exercise 8 from Sections 14-16, page 92

In order to avoid ambiguity between the lower limit and upper limit topologies, we will choose an orientation for L. If L is parallel to either axis, choose its positive direction to agree with that of the axis it is parallel to. This gives immediately that the topology it inherits as a subspace is that of the corresponding axis: in $\mathbb{R}_\ell \times \mathbb{R}$, if L is parallel to the y-axis, it inherits the usual topology; in any of the other three cases, it inherits the lower limit topology. This settles the trivial cases.

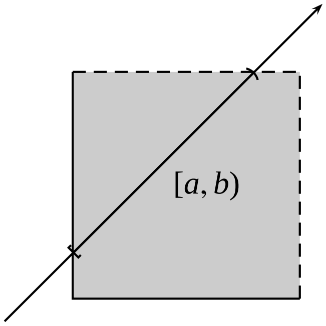
If L is not parallel to either axis, choose its positive direction to agree with the positive direction of the x-axis. This choice ensures that we'll always get the lower (rather than upper) limit topology. Let's consider $\mathbb{R}_\ell \times \mathbb{R}$ first. The basic open sets are of the form $[a, b) \times (c, d)$, and their possible intersections with L are shown below for L with positive and negative gradients:

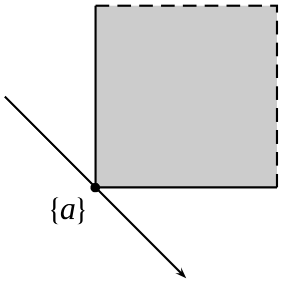




In either case, the subsets of L we get are of the form $[a, b)$ or (a, b). Since sets of the form (a, b) can be obtained by taking unions of sets of the form $[a, b)$, we see that the topology on L can be generated by the basis $\{[a, b) \mid a < b\}$, i.e. L has the lower limit topology.

Now for $\mathbb{R}_\ell \times \mathbb{R}_\ell$. The basic open sets for $\mathbb{R}_\ell \times \mathbb{R}_\ell$ can be chosen to be $[a, b) \times [c, d)$, and their intersections with L are as follows:





For L with positive gradient, the only subsets we get are of the form $[a.b)$, so as before, L gets the lower limit topology. For L with negative gradient, it is possible that the intersection of a basic open set with L gives a singleton set. Hence all singleton subsets of L are open, and L inherits the discrete topology.

To summarize, in $\mathbb{R}_\ell \times \mathbb{R}$, L inherits the lower limit topology except when it is parallel to the y-axis, and in that case it inherits the usual topology on $\mathbb{R}$. For $\mathbb{R}_\ell \times \mathbb{R}_\ell$, L gets the lower limit topology except when it has negative gradient, and in that case it has the discrete topology. □

Exercise 9 from Sections 14-16, page 92

Let $\mathscr{T}_o$, $\mathscr{T}_d$ and $\mathscr{T}_s$ denote the dictionary order, product of the discrete and standard topologies, and the standard topology on $\mathbb{R} \times \mathbb{R}$, respectively. We will use the following bases for these topologies:

$$\begin{array}{ll} \mathscr{T}_o: & \mathscr{B}_o = \{(a \times b, c \times d) \mid (a < c) \vee (a = c \wedge b < d)\} \\ \mathscr{T}_d: & \mathscr{B}_d = \{\{a\} \times (b, c) \mid b < c\} \\ \mathscr{T}_s: & \mathscr{B}_s = \{(a, b) \times (c, d) \mid (a < b) \wedge (c < d)\}. \end{array}$$

To show that $\mathscr{T}_o = \mathscr{T}_d$, it suffices to check that $\mathscr{B}_o \subset \mathscr{T}_d$ and $\mathscr{B}_d \subset \mathscr{T}_o$. First observe that $\{a\} \times (b, c)$ is precisely the interval $(a \times b, a \times c)$ in the dictionary order. This gives the second inclusion, as well as $(a \times b, a \times c) \in \mathscr{T}_d$. The only case left to consider is whether intervals of the form $(a \times b, c \times d)$, $a < c$ are in $\mathscr{T}_d$. The answer is affirmative, as they can be written as

$$(a \times b, c \times d) = \{a\} \times (b, \infty) \cup \bigcup_{x \in (a,c)} \{x\} \times \mathbb{R} \cup \{c\} \times (-\infty, d),$$

and $\{x\} \times \mathbb{R}$ can be expressed as $\{x\} \times \mathbb{R} = \bigcup_{n \in \mathbb{Z}} \{x\} \times (n, n+1)$, and similarly for the rays (b, ∞) and $(-\infty, d)$. Hence $(a \times b, c \times d)$ is a union of elements of $\mathscr{B}_d$.

Finally, we show that $\mathscr{T}_s \subsetneq \mathscr{T}_d$. To show inclusion, we use the same trick as above, writing

$$(a, b) \times (c, d) = \bigcup_{x \in (a,b)} \{x\} \times (c, d).$$

To see that the inclusion is strict, note that $\{a\} \times (c, d)$ is not in $\mathscr{T}_s$ since $\pi_1(\{a\} \times (c, d)) = \{a\}$ is not open in $\mathbb{R}$ (with the standard topology). □