Math 453
Solutions to Assignment 2
September 20, 2004

Exercise 1 from Sections 12 & 13, page 83

We will show that A is open by exhibiting it as a union of open sets. For each $x \in A$, let U_x be the open set containing x such that $U_x \subset A$. It is easy to see that $A = \bigcup_{x \in A} U_x$, so A is open. $\square$

Exercise 4 from Sections 12 & 13, page 83

(a) Let $\mathscr{T} = \bigcap \mathscr{T}_\alpha$. To show that $\mathscr{T}$ is a topology, we have to verify that $\mathscr{T}$ satisfies the three properties in the definition of a topology:

(1) Are $\varnothing$ and X in $\mathscr{T}$? Yes, because $\varnothing$ and X are in $\mathscr{T}_\alpha$ for each α.

(2) Let $\{U_\beta\}$ be a collection of open sets in $\mathscr{T}$. Since $\mathscr{T}$ is the intersection of the topologies $\mathscr{T}_\alpha$, $\{U_\beta\}$ is a collection of open sets in $\mathscr{T}_\alpha$ for each α. Hence their union $\bigcup U_\beta$ is in $\mathscr{T}_\alpha$ for each α, and so $\bigcup U_\beta \in \mathscr{T}$.

(3) Starting with a finite collection of open sets in $\mathscr{T}$, the argument is as in (2) above.

This proof works for the intersection $\bigcap \mathscr{T}_\alpha$ because subsets in the intersection are open in each $\mathscr{T}_\alpha$. For the union of even two topologies, say $\mathscr{T}_1$ and $\mathscr{T}_2$, we can have subsets $U_1 \in \mathscr{T}_1$ and $U_2 \in \mathscr{T}_2$ such that $U_1 \cup U_2$ is in neither $\mathscr{T}_1$ nor $\mathscr{T}_2$. A simple example is furnished by the topologies $\mathscr{T}_1 = \{\varnothing, X, \{b\}\}$ and $\mathscr{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}$ on the three point set $X = \{a, b, c\}$. Taking $U_1 = \{b\}$ and $U_2 = \{a\}$, their union $\{a, b\}$ is not in either of $\mathscr{T}_1$ or $\mathscr{T}_2$.

(c) Let $\mathscr{T}$ be the topology on X generated by the subbasis $\mathscr{T}_1 \cup \mathscr{T}_2$. It is easily checked that $\mathscr{T}$ is the smallest topology containing both $\mathscr{T}_1$ and $\mathscr{T}_2$. In this particular case,

$$\mathscr{T}_1 \cup \mathscr{T}_2 = \{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}\}$$

and the only other subset that can be generated by taking unions of finite intersections is $\{b\}$. Hence the smallest topology containing both $\mathscr{T}_1$ and $\mathscr{T}_2$ is

$$\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$$

The largest topology contained in both $\mathscr{T}_1$ and $\mathscr{T}_2$ is clearly their intersection, so it is

$$\mathscr{T}_1 \cap \mathscr{T}_2 = \{\varnothing, X, \{a\}\}$$

in this case.

□

Exercise 8 from Sections 12 & 13, page 83

(a) Let U be an open set in $\mathbb{R}$ and x an element in U. By the definition of the standard topology on $\mathbb{R}$, there are *real* numbers r, s such that $x \in (r, s) \subset U$. Since the rationals are dense in $\mathbb{R}$, we can find rational numbers a, b in (r, x) and (x, s), respectively. This gives $x \in (a, b) \subset U$. In other words, given U open in $\mathbb{R}$ and $x \in U$, we can find an $(a, b) \in \mathscr{B}$ such that $x \in (a, b) \subset U$. Hence, by Lemma 13.2, $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}$.

(b) Given $x \in \mathbb{R}$, there are certainly rational numbers a, b such that $a < x < b$, so $x \in [a, b)$. Thus $\mathscr{C}$ satisfies condition (1) for a basis. For condition (2), simply note that the intersection of two intervals of the form $[a, b)$ is either empty or another interval of the same form. Hence $\mathscr{C}$ is a basis for a topology on $\mathbb{R}$; call this topology $\mathscr{T}$.

Each element of $\mathscr{C}$ is open in the lower limit topology, so $\mathscr{T}$ is contained in the lower limit topology. To see that they are different, consider the open set $[r, s)$ in $\mathbb{R}_\ell$, where r is irrational. If $[r, s)$ were open in $\mathscr{T}$, then, since $\mathscr{C}$ is a basis for $\mathscr{T}$ and $r \in [r, s)$, there must be $a, b \in \mathbb{Q}$ such that $r \in [a, b) \subset [r, s)$. This is clearly impossible for rational a, so $[r, s)$ cannot be in $\mathscr{T}$.

□