MATH 453

SOLUTIONS TO ASSIGNMENT 1

SEPTEMBER 8, 2004

Exercise 4 from Section 3, page 28

(a) Reflexivity, symmetry and transitivity for $\sim$ follow from the corresponding properties for equality, so $\sim$ is an equivalence relation. For instance, to check symmetry, we proceed as $(a_0 \sim a_1) \Rightarrow (f(a_0) = f(a_1)) \Rightarrow (f(a_1) = f(a_0)) \Rightarrow (a_1 \sim a_0)$.

(b) Let $A^* = \{[a] \mid a \in A\}$ be the equivalence classes under $\sim$. The map $f : A \to B$ induces a map $f^* : A^* \to B$ given by $f^*([a]) = f(a)$. We will check that f^* is a bijection.

Given $b \in B$, there is an $a \in A$ such that $f(a) = b$ since f is surjective. Hence $f^*([a]) = b$ and so f^* is surjective. For injectivity, suppose $f^*([a_0]) = f^*([a_1])$. By the definition of f^* we get $f(a_0) = f(a_1)$, or $a_0 \sim a_1$. Hence $[a_0] = [a_1]$ and f^* is injective.

□

Exercise 5 from Section 3, page 28

(a) Again we have to check the three conditions for an equivalence relation. For reflexivity, note that $x - x = 0$, so $(x, x) \in S'$. If $y - x$ is an integer, then $x - y = -(y - x)$ is too, and symmetry follows. Finally, if $y - x$ and $z - y$ are integers, then so is $z - x = (z - y) + (y - x)$. Hence $(x, z) \in S'$ and we have transitivity.

If $(x, y) \in S$, then $y = x + 1$, or $y - x = 1$, an integer. Hence $(x, y) \in S'$. This shows that $S' \supset S$.

Given $x \in \mathbb{R}$, the equivalence class $[x]$ consists of all $y \in \mathbb{R}$ such that $y - x \in \mathbb{Z}$. In other words, y is of the form $x + n$ where $n \in \mathbb{Z}$. Denoting the set $\{x + n \mid n \in \mathbb{Z}\}$ by $x + \mathbb{Z}$, we have $[x] = x + \mathbb{Z}$. Furthermore, given any $x \in \mathbb{R}$, there is an $x' \in [0, 1)$ and an integer n such that $x = x' + n$. Hence $[x] = [x']$ for some $x' \in [0, 1)$. Also, no two distinct elements of $[0, 1)$ are S'-related, so $\mathbb{R}/S' = \{x + \mathbb{Z} \mid x \in [0, 1)\}$.

□

Exercise 5 from Section 5, page 39

Before looking at the specific subsets in this problem, let us consider an arbitrary subset $A = \{\mathbf{x} \mid \mathbf{x} \text{ satisfies some condition } P\}$. Suppose P does not mix the indices, i.e. it is of the form "for all i, x_i satisfies some condition $C(i)$ that depends only on i". This means that the ith coordinate x_i can only take values from the subsets A_i of $\mathbb{R}$ given by $A_i = \{x \mid x \text{ satisfies } C(i)\}$. Hence A is simply the product of the subsets A_i. With this in mind, we see that subsets in (a), (b) and (c) can be expressed as products of subsets of $\mathbb{R}$, namely:

(a) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i\} = \prod_{i \in \mathbb{Z}_+} A_i$ where $A_i = \mathbb{Z}$ for all i.

(b) $\{\mathbf{x} \mid x_i \geq i \text{ for all } i\} = \prod_{i\in\mathbb{Z}_+} A_i$ where $A_i = [i, \infty)$ for all i.

(c) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i \geq 100\} = \prod_{i\in\mathbb{Z}_+} A_i$ where

$$A_i = \begin{cases} \mathbb{R} & \text{if } 1 \leq i < 100 \\ \mathbb{Z} & \text{if } i \geq 100. \end{cases}$$

On the other hand, if P does mix the indices, then it is in general not possible to write A as a product of subsets of $\mathbb{R}$.

(d) Let $B = \{\mathbf{x} \mid x_2 = x_3\}$. We will show that B cannot be a product of subsets of $\mathbb{R}$ by assuming that it is and arriving at a contradiction. Suppose B is a product $\prod_{i\in\mathbb{Z}_+} B_i$ of subsets B_i of $\mathbb{R}$. First note that elements of B are of the form $(x_1, x_2, x_3 = x_2, x_4, \dots)$, and B_i must contain all values that x_i can take. Since there are no restrictions on what values x_i, $i \neq 3$ can take, we must have $B_i = \mathbb{R}$ for $i \neq 3$. Also, x_3, being a mirror of x_2, can be any real number, so $B_3 = \mathbb{R}$ too. This implies that $B = \mathbb{R}^\omega$, which is a contradiction since $(0, 0, 1, 0, 0, \dots)$ is clearly not in B.

□

Exercise 4 from Section 7, page 51

(a) Let P_n be the set of monic polynomials of degree n. Then there is a bijection between P_n and $\mathbb{Q}^n$ given by

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \leftrightarrow (a_{n-1}, a_{n-2}, \dots, a_0).$$

Thus P_n is countable. Since each polynomial has only finitely many roots, the set of real numbers that satisfy a monic polynomial of degree n (call it $\mathbb{A}_n$) is countable. Finally, the set of algebraic numbers $\mathbb{A}$ is the countable union $\mathbb{A} = \bigcup_{n=1}^{\infty} \mathbb{A}_n$ and so is countable.

(b) If $\mathbb{T}$ denotes the transcendental numbers, then $\mathbb{R} = \mathbb{A} \cup \mathbb{T}$. Hence $\mathbb{T}$ is uncountable, least $\mathbb{R}$ be countable.

□

Exercise 5 from Section 7, page 51

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$ is just the Cartesian product $\mathbb{Z}_+ \times \mathbb{Z}_+$, so it is countable.

(b) Similarly, the set B_n of all functions $f : \{1, \dots, n\} \to \mathbb{Z}_+$ is the Cartesian product $\underbrace{\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+}_{n \text{ copies}}$. Hence B_n is also countable.

(c) $C = \bigcup_{n\in\mathbb{Z}_+} B_n$ is a countable union of countable sets, so is countable.

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is uncountable. The proof is essentially the same as that for Theorem 7.7 in the textbook; we just need to choose a different $\mathbf{y} = (y_1, y_2, \dots, y_n, \dots)$, say $y_n = x_{nn} + 1$.

(e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ is the same as $\{0, 1\}^\omega$ from Theorem 7.7. Thus it is uncountable.

□

Exercise 6 from Section 7, page 51

(a) Suppose $B \subset A$ and there is a injective function $f : A \to B$. We want to construct a bijection $h : A \to B$. Following the hint, we define $A_1 = A$, $B_1 = B$ and $A_n = f(A_{n-1})$, $B_n = f(B_{n-1})$ for $n > 1$. Then $A_1 \supset B_1 \supset A_2 \supset B_2 \supset \cdots$. Define, in addition, the following sets:

$$R_n = A_n - B_n; R = \bigcup_{n=1}^{\infty} R_n, R' = \bigcup_{n=2}^{\infty} R_n \quad \text{and} \quad S_n = B_n - A_{n+1}; S = \bigcup_{n=1}^{\infty} S_n.$$

Note that R and S are disjoint, $A = R \cup S$ and $B = R' \cup S$.

The function $h : A \to B$ given by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise} \end{cases}$$

is simply the union of the two functions $f|_R : R \to R'$ and $id_S : S \to S$, where $f|_R$ is the restriction of f to R and id_S is the identity function on S. Thus h is a bijection if and only if $f|_R$ and id_S are. id_S, being the identity on S, is certainly bijective, so we need only consider $f|_R$.

Since f is injective, so is its restriction $f|_R$. To show surjectivity, let us examine what f does to the (disjoint) sets R_n that make up R. I claim that $f(R_n) = R_{n+1}$. R' is just R sans R_1, so this claim implies the surjectivity of $f|_R$.

Now to prove the claim. Suppose $x \in R_n$. Then x is in A_n but not in B_n. Thus $f(x) \in A_{n+1}$. Is $f(x)$ in B_{n+1}? No, because f is injective and x is not in B_n. This shows that $f(R_n) \subset R_{n+1}$. If $y \in R_{n+1}$, then $y \in A_{n+1}$ but $y \notin B_{n+1}$. This means there is an $x \in A_n$ such that $f(x) = y$. Can x be in B_n? No, for if it does, then $y = f(x) \in B_{n+1}$. Hence $x \in A_n - B_n$, that is, $x \in R_n$.

(b) Suppose we are given injections $f : A \to C$ and $g : C \to A$. First note that $f : A \to f(A)$ is a bijection, so A and $f(A)$ have the same cardinality. Now the relation "have the same cardinality" is clearly an equivalence relation, so it suffices to show that $f(A)$ and C have the same cardinality. This follows from part (a) as $f \circ g : C \to f(A)$ is a composition of two injections, and so is an injection.

□

Exercise 7 from Section 7, page 52

We will construct injections $\varphi : D \to E$ and $\psi : E \to D$ and use the Schroeder-Bernstein Theorem to conclude that D and E have the same cardinality.

ψ is easy: given $f \in E$, that is, a function from $\mathbb{Z}_+$ to $\{0, 1\}$, define $\psi(f)$ to be the function that assigns to n the value $f(n) + 1$. Then $\psi(f) : \mathbb{Z}_+ \to \{1, 2\} \subset \mathbb{Z}_+$. ψ is injective because $\psi(f) = \psi(g)$ implies $f(n)+1 = g(n)+1$ for all n, so $f(n) = g(n)$ for all n, i.e. $f = g$.

To construct φ it will be more convenient to use the language of sequences. Recall that a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ can be written as a sequence of positive integers. Similarly, a function $g : \mathbb{Z}_+ \to \{0, 1\}$ is a sequence of 0's and 1's. In this perspective, what we need to do is take a sequence of positive integers and produce a binary sequence, and ensure that difference sequences of positive integers always give different binary sequences.

One way to do this can be found by examining how we usually write sequences. We write the first element, a delimiter (usually a comma), then the next element, a delimiter, and so on: $a_1, a_2, a_3, \ldots$. If we use 1 as the delimiter and the number of 0's to represent the elements a_i, we have

$$\varphi((a_1, a_2, a_3, \ldots)) = (\underbrace{0, \ldots, 0}_{a_1 \text{ zeros}}, 1, \underbrace{0, \ldots, 0}_{a_2 \text{ zeros}}, 1, \underbrace{0, \ldots, 0}_{a_3 \text{ zeros}}, 1, \ldots).$$

To see that φ is injective, observe that given $\varphi((a_1, a_2, \ldots))$, we can recover the numbers a_i: a_1 is the number of 0's before the first 1, a_2 is the number of 0's between the first and second 1's, and so on. Thus given any binary sequence, there is at most one element of D that gets mapped to it by φ.

There are, of course, many other possible injective functions $\varphi : D \to E$. Have fun devising your own. □