The Birman-Craggs-Johnson homomorphism and abelian cycles in the Torelli group

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Abstract

We use the Birman-Craggs-Johnson homomorphism and the method of abelian cycles to give lower bounds on the rank of the second homology of the Torelli group $\mathcal{I}$ and of its subgroup the Johnson kernel $\mathcal{K}$ with coefficients in $\mathbf{Z}/2\mathbf{Z}$.

1 Introduction

Let $S = S_{g,b}$ denote a closed, oriented surface of genus g with b boundary components, and let $\mathrm{Mod}_{\mathrm{g,b}}$ denote its mapping class group, which is the group of homotopy classes of orientation-preserving homeomorphisms preserving the boundary components pointwise. For the purposes of this paper, we will always assume that $b \in \{0, 1\}$.

The *Torelli group* of $S_{g,b}$, denoted $\mathcal{I} = \mathcal{I}(S_{g,b})$, is the subgroup of $\mathrm{Mod}_{\mathrm{g,b}}$ which acts trivially on $H = H_1(S_{g,b})$. We denote by $\mathcal{K}$ the subgroup of $\mathcal{I}$ generated by Dehn twists about separating curves on $S_{g,b}$.

Facts about $\mathcal{K}$: Morita, Biss-Farb, Brendle-Margalit, Johnson Structure 1, Mess

Though the fundamental question of finite generation has been settled now in the affirmative for $\mathcal{I}$ [Jo2] and in the negative for $\mathcal{K}$ [BF], the question of whether $\mathcal{I}$ admits a finite presentation remains open. Another gap in our understanding of $\mathcal{I}$ and $\mathcal{K}$ is we do not know how to compute the

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(co)homology groups of both $\mathcal{I}$ and $\mathcal{K}$, with the only exception being Johnson's calculation of $H_1(\mathcal{I}, \mathbf{Z})$ [Jo5]. The degree 1 case is unknown for $\mathcal{K}$, and in light of the recent result of Biss and Farb, it would be interesting to see whether $H_1(\mathcal{K}, \mathbf{Z})$ is finitely generated (a question posed by Biss and Farb in [BF]). Further, the open problem of calculating (co)homology is related to that of finite presentation of $\mathcal{I}$: if $H_2(\mathcal{I}, G)$ is not finitely generated, then $\mathcal{I}$ itself does not admit a finite presentation (REF – Ken's book??).

We will begin with a brief overview of what is known about the (co)homology of $\mathcal{I}$....

Rational (co)homology of $\mathcal{I}$. Prior to this work, virtually all our knowledge of $H_*(\mathcal{I})$, $H^*(\mathcal{I})$ came from the *Johnson homomorphism*

$$\tau : \mathcal{I} \to \wedge^3 H$$

which arises from the action of the Torelli group on certain nilpotent quotients of $\pi_1(S)$. (Johnson first defined the map τ in [JoAb], but we also refer the reader to Johnsons's survey article [Jo5] for several equivalent definitions of τ.)

We note three important facts about the map τ: (1) τ is surjective onto $\wedge^3 H$, (2) τ is equivariant with respect to the natural action of $\mathrm{Mod}(S)$ on both $\mathcal{I}$ and $\wedge^3 H$, and (3) the kernel of τ is precisely the group $\mathcal{K}$ [Jo3]; hence $\mathcal{K}$ is sometimes referred to as the *Johnson kernel.*

Consider the induced map on cohomology with rational coefficients:

$$\tau^* : H^*(\wedge^3 H, \mathbf{Q}) \to H^*(\mathcal{I}, \mathbf{Q})$$

The map τ^* is fairly well understood in degrees 1 and 2. In degree 1, Johnson proved that τ^* is an isomorphism. In other words, the image of τ^* contains all rational abelian information about the Torelli group. In degree 2, Hain computed the kernel of τ^* using techniques of representation theory and algebraic geometry [Hai]. Further, in degree 3, Sakasai has computed $\ker(\tau^*)$ up to one irreducible summand using methods similar to Hain's [Sa].

This approach, while fruitful in many respects, necessarily sacrifices information on two points: the subgroup $\mathcal{K} = \ker \tau$, and torsion. While $\mathcal{I}$ itself has no torsion, its abelianization does. More precisely, Johnson computed in [Jo4]:

$$H_1(\mathcal{I}, \mathbf{Z}) \cong \wedge^3 H \oplus B_2$$

where B_2 is isomorphic as an abelian group to a certain finite number of copies of $\mathbf{Z}/2\mathbf{Z}$. (See [vdB] for a beautiful new approach to Johnson's result.) In fact, B_2 possesses a bit more structure, which we now describe.

Boolean polynomials. We define $B(x_1, \ldots, x_n)$, the ring of *Boolean* (or *square-free*) polynomials, to be the quotient of the usual polynomial ring on the variables $x_1, \ldots, x_n$ with coefficients in $\mathbf{Z}/2\mathbf{Z}$ by the ideal generated by the relations $x_i^2 = x_i$ for all $i = 1, \ldots, n$. In other words

$$B(x_1, \ldots, x_n) = \mathbf{Z}_2[x_1, \ldots, x_n] / < x_i^2 = x_i >$$

Then we let B_r denote the subset of all Boolean polynomials in the given variables of degree less than or equal to r. Thus we can think of B_r as a $\mathbf{Z}/2\mathbf{Z}$-vector space of dimension $\sum_{i=0}^{2} \binom{n}{i}$, with basis $\{1, x_i, x_j x_k\}$ where $1 \leq i \leq n$ and $1 \leq j < k \leq n$. For our purposes, we shall take $n = 2g$, where g is the genus of our surface. Later we shall attach a certain topological significance to these $2g$ variables.

The Birman-Craggs-Johnson homomorphism. Birman and Craggs defined a family of surjections $\mathcal{I} \to \mathbf{Z}/2\mathbf{Z}$ arising from the Rochlin invariant of spin 3-manifolds [BC]. Building on their work, Johnson constructed the *Birman-Craggs-Johnson (BCJ) homomorphism* [Jo1]:

$$\sigma : \mathcal{I} \to B_3$$

The map σ encodes all of the Birman-Craggs homomorphisms in the sense that the kernel of σ is the intersection of the kernels of all Birman-Craggs homomorphisms. As with Johnson's homomorphism τ, $Mod(S)$ acts naturally on both $\mathcal{I}$ and the target of the map, and σ respects this action.

Johnson showed that, unlike the case of the map τ, the restriction of σ to $\mathcal{K}$ is highly nontrivial. In fact, he showed that:

$$\sigma|_{\mathcal{K}} : \mathcal{K} \to B_2$$

is a surjection; in other words, $\sigma(\mathcal{K})$ is precisely equal to $B_2 \subset B_3$. Thus we can hope to gain information about both 'missing pieces' from the Hain-Sakasai approach via the BCJ homomorphism.

Main Results. In this paper, we will use the BCJ homomorphism σ together with the method of abelian cycles to give an explicit construction of the first known nontrivial classes in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$. Further, these classes also survive in $H_2(\mathcal{K}, \mathbf{Z}/2/Z)$

We will use the induced map on second homology

$$\sigma_* : H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}) \to H_2(B_2, \mathbf{Z}/2\mathbf{Z})$$

to characterize these classes. Applying the Universal Coefficient Theorem, we have that

$$H_2(B_2, \mathbf{Z}/2\mathbf{Z}) \cong \wedge^2(B_2) \oplus B_2$$

(We give the details of this calculation are given in what is currently the Appendix.) Thus we have:

$$\sigma_* : H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}) \to \wedge^2 B_2 \oplus B_2 \tag{1}$$

Though a precise statement must wait until later, after we present the necessary background, our results can be summarized by saying we can hit most of the $\wedge^2(B_2)$ summand in $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$:

Main Theorem. *The image of* σ_* *contains all but 2* $\mathrm{Mod}(S)$*-orbits of basis elements of the summand* $\wedge^2(B_2) \subseteq H_2(B_2, \mathbf{Z}/2\mathbf{Z}))$.

We shall give an explicit list of the classes in $H_2(B_2, \mathbf{Z}/2\mathbf{Z}))$ hit by σ_*, and we obtain a lower bound on the rank of $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}))$:

Corollary 1. *The rank of* $H_2(\mathcal{I}(S), \mathbf{Z}/2\mathbf{Z})$ *is at least on the order of* g^4.

Moreover, it follows from our construction that each of these classes survives in the full Torelli group; in other words, we have:

Corollary 2. *The rank of* $H_2(\mathcal{I}(S), \mathbf{Z}/2\mathbf{Z})$ *is at least on the order of* g^4.

We shall see that the best possible result one could hope to achieve with the method of abelian cycles would be that the image of σ_* contains all of $\wedge^2 B_2$, but certain topological obstructions arise in the application of our methods which prevent us from hitting certain classes in $\wedge^2 B_2 \subset H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}))$. (Do I even want to say this up front???)

B. -Hain, Sakasai method ($Sp - \mathbf{Q}$ - rep theory)

-In our case we can't use modular representations

-What we do: Find abelian cycles (define, examples) and evaluate (discuss briefly $\cup$ vs. $\cap$, etc.)

2 Sp-quadratic forms and Johnson's formula for σ

Birman-Craggs homomorphisms. Birman and Craggs produced the first known abelian quotients of the Torelli group by defining a collection of surjective maps $\mathcal{I} \to \mathbf{Z}/2\mathbf{Z}$. [BC]. Their construction used the Rochlin invariant, or μ-invariant, of (spin) 3-manifolds, first introduced in [EK] and defined as follows. Let M be an oriented 3-manifold with a given spin structure. Choose X, a spin 4-manifold such that $\partial X = M$ and such that this restriction to the boundary induces the given spin structure on M. Then the Rochlin invariant μ of M is given by

$$\mu(M) = \frac{\sigma(X)}{8} \mod 2.$$

This expression is independent of the choice of the 4-manifold X.

For the sake of completeness, we recall the following facts about the Rochlin invariant μ of oriented (spin) 3-manifolds, first defined in [EK].

- If M is a $\mathbf{Z}/2\mathbf{Z}$-homology sphere, then $\mu(M) \in \frac{1}{4}\mathbf{Z} \bmod 2\mathbf{Z}$.
- If M is a $\mathbf{Z}$-homology sphere, then $\mu(M) \in \mathbf{Z}/2\mathbf{Z}$.
- $\mu(S^3) = 0$.

We give the definition of the Birman-Craggs homomorphisms as reformulated by Johnson in [Jo5]. Let

$$h : S_{g,0} \to S^3$$

be an embedding, and let $f \in \mathcal{I}(S_{g,0}$. Now split S^3 along $h(S_{g,0})$ and reglue via the map f. The resulting 3-manifold $M(h,f)$ is necessarily a homology 3-sphere. We therefore define the *Birman-Craggs homomorphism associated to the embedding h* as follows:

$$\begin{aligned} \rho_h : \mathcal{I}(S_{g,0}) &\to \mathbf{Z}_2 \\ f &\mapsto \mu(M(h,f)) \end{aligned}$$

Birman and Craggs proved that although such homomorphisms are a priori indexed by a infinite set, there are in fact only finitely many distinct Birman-Craggs homomorphisms. (We note that Birman and Craggs originally defined the maps ρ_h in terms of pairs of maps h_1, h_2 such that the result of gluing two handlebodies together along their common boundary via the product $h_1 h_2$ is a $\mathbf{Z}_2$-homology sphere.) Birman and Craggs then posed the problem of enumerating distinct Birman-Craggs homomorphisms.

Johnson answered their question in [Jo1] by showing that the Birman-Craggs homomorphisms ρ_h are in one-to-one correspondence with (mod 2) self-linking forms on $H_1(S_{g,0})$. Given an embedding h of the oriented surface $S_{g,0}$ in S^3, the corresponding mod 2 self-linking form $\omega_h : H_1(S_{g,0}) \to \mathbf{Z}/2\mathbf{Z}$

is defined on irreducible elements $c \in H_1(S_{g,0})$ by $\omega_h(c) = lk(h(c), h(c)^+)$, where the latter expression denotes the linking number of a representative of the homology class $h(c)$ with its positive push-off $h(c)^+$ in S^3. For simplicity, we let $H = H_1(S_{g,0}, \mathbf{Z}/2\mathbf{Z})$.

All mod 2 self-linking forms are functions $\omega : H \to \mathbf{Z}/2\mathbf{Z}$ whose associated bilinear form is just the usual symplectic intersection pairing on H. Let Ω denote the set of all such functions. Thus

$$\Omega = \{\omega : H \to \mathbf{Z}/2\mathbf{Z} | \quad \omega(a+b) = \omega(a) + \omega(b) + a \cdot b\}$$

where $a \cdot b$ denotes the intersection form. One of Johnson's main results in [Jo1] is that mod 2 self-linking forms, and hence the set of distinct Birman-Craggs homomorphisms, are in one-to-one correspondence with a particular subset of Ω, which we will now describe.

Let $a_1, b_1, \ldots, a_g, b_g$ be a fixed symplectic basis for H. Now for each $c \in H$, we can define a map

$$\begin{aligned} \overline{c} : \Omega &\to \mathbf{Z}/2\mathbf{Z} \\ \omega &\mapsto \omega(c). \end{aligned}$$

We will need two basic facts which follow directly from the definitions:

- for all $a, b \in H$, we have $\overline{a+b} = \overline{a} + \overline{b} + a \cdot b$
- for all $c \in H$, $\overline{c}^2 = \overline{c}$.

We can now form Boolean polynomials in the obvious way out of elements $\overline{c}$, with $c \in H$. Specifically, we take as abstract variables the $2g$ maps $\overline{a}_1, \ldots, \overline{a}_g, \overline{b}_1, \ldots \overline{b}_g$. As above, we denote the ring of Boolean polynomials of degree less than or equal to r in these $2g$ variables simply by B_r. As a $\mathbf{Z}/2\mathbf{Z}$-vector space, B_r has dimension $\sum_{i=0}^{r} \binom{2g}{i}$. For example, a basis for B_2 is given by

$$\{1, \overline{a}_i, \overline{b}_i, \overline{a}_i\overline{b}_j, \overline{a}_i\overline{a}_j (i \neq j), \overline{b}_i\overline{b}_j (i \neq j)\}.$$

We single out the *Arf invariant*, given by

$$\alpha = \sum_{i=1}^{g} \overline{a}_i \overline{b}_i$$

as an example of a quadratic Boolean polynomial. (PERHAPS SAY SOMETHING HERE about the role the Arf invariant plays in knot theory and in

3-manifolds???) Let Ψ denote the subset of Ω consisting of those Sp-forms with zero Arf invariant. In other words,

$$\Psi = \{\omega \in \Omega | \alpha(\omega) = 0\}.$$

Johnson proves in [Jo1] that the Birman-Craggs homomorphisms are in one-to-one correspondence with elements of Ψ and that the order of Ψ for surface of genus g is $2^{2g-1}+2^{g-1}$. It is this bijection which enables Johnson to formulate appropriate and analogous definitions for a BCJ homomorphism defined on $\mathcal{I}_{g,1}$ (I WANT TO SAY A BIT MORE HERE to clarify the closed surface vs. one boundary component issue).

Further, Johnson poses and answers the question of describing the space of Birman-Craggs homomorphisms. His solution, essentially, is to gather all Birman-Craggs homomorphisms together into one, the *Birman-Craggs-Johnson* (BCJ) homomorphism:

$$\sigma : \mathcal{I}_{g,1} \to B_3$$

As previously noted, we have $\ker\sigma = \cap_h \ker\rho_h$, where the intersection is taken over all Birman-Craggs homomorphisms.

Johnson's formula. Johnson gives an explicit formula for $\sigma(f)$, where $f \in \mathcal{I}_{g,1}$ is either a twist about a separating curve or else a BP-map, in terms of a symplectic homology basis for the subsurface of $S_{g,1}$ bounded by the defining curve(s) of the map f. Since BP-maps generate $\mathcal{I}_{g,1}$ [Jo2] and twists about separating curves generates $\mathcal{K}$ by definition, we can take Johnson's formula as the definition of the BCJ homomorphism σ, or of its restriction $\sigma|_{\mathcal{K}}$.

Let $T_\gamma T_\delta^{-1}$ be a BP-map which bounds a subsurface $S_{\gamma\delta} \subset S_g$. Let $A_1, B_1, \ldots, A_{g(S')}, B_{g(S')}$ be a symplectic Z_2-homology basis for S', and let C be the homology class of γ (orient γ so that S' is on its left). Then we define the BCJ homomorphism

$$\begin{array}{rccc} \sigma : & \mathcal{I}_g & \to & B_3 \\ & T_\gamma T_\delta^{-1} & \mapsto & \left(\sum_{i=1}^{g(S_{\gamma\delta})} \overline{A}_i \overline{B}_i\right)(\overline{C}+1) \end{array}$$

The expression on the right-hand side of the equation is independent of the choice of symplectic basis for $S_{\gamma\delta}$.

Similarly, for T_α, a twist about a separating curve α, we can write

$$\begin{array}{rccc} \sigma|_{\mathcal{K}}: & \mathcal{K} & \to & B_2 \\ & T_\alpha & \mapsto & \left(\sum_{i=1}^{g(S_\alpha)} \overline{A}_i \overline{B}_i\right) \end{array}$$

where S_α is the subsurface of $S_{g,1}$ bounded by α and $\{A_i, B_i\}$ is a symplectic basis for S_α.

Both maps are surjective.

Remark. As previously noted, B_3 is an $\mathrm{Mod}_{g,1}$-module in a fairly obvious way. We have that $f \in Sp(2g, \mathbf{Z}_2)$ acts on a map $\omega : H \to \mathbf{Z}_2$ in Ω by $f \cdot \omega(x) = \omega(f(x))$. Then Sp acts on a function $\phi : \Omega \to \mathbf{Z}_2$ adjoint to its action on Ω, i.e., $f \cdot \phi(\omega) = \phi(f \cdot \omega)$. Furthermore, σ is a $\mathrm{Mod}_{g,1}$-equivariant map. Specifically, for f in the mapping class group and $g \in T_g$, $\sigma(fgf^{-1}) = \tilde{f} \cdot \sigma(g)$, where $\tilde{f}$ denotes the image of the map f under the symplectic representation mod 2.

Moreover, since $\mathcal{K}$ is an $\mathrm{Mod}_{g,1}$-submodule of $\mathcal{I}$, we have that $\sigma(\mathcal{K}) = B_2$ is also an $\mathrm{Mod}_{g,1}$-module and that $\sigma|_{\mathcal{K}}$ is also a $\mathrm{Mod}_{g,1}$-equivariant map.

As we see, the formula for a generator of $\mathcal{K}$ is a bit simpler than for a BP generator of $\mathcal{I}$. Because of its relative simplicity, we shall now restrict our attention to the subgroup $\mathcal{K}$ in $\mathcal{I}$ and write simply

$$\sigma : \mathcal{K} \to B_2 \tag{2}$$

for the BCJ homomorphism. However, the tools and techniques we are about to discuss will go through in the case of the full Torelli group.

3 Abelian cycles

In this section, we will give a method for constructing nontrivial classes in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ inspired by Hain, Sakasai, (Pitsch?). The idea is to construct abelian cycles in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ and then to show that their images under the induced map

$$\sigma_* : H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}) \to H_2(B_2, \mathbf{Z}/2\mathbf{Z}) \cong \wedge^2 B_2 \oplus B_2$$

are nontrivial.

Abelian cycles Let $f, g \in \mathcal{K}$ be two maps which commute. The Torelli group has no torsion; thus $\langle f, g\rangle \cong \mathbf{Z} \times \mathbf{Z}$ and we have an injection

$$i : \mathbf{Z} \times \mathbf{Z} \to \mathcal{K}$$

which induces a map on second homology:

$$i_* : H_2(\mathbf{Z} \times \mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \to H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$$

Since $H_2(\mathbf{Z} \times \mathbf{Z}, \mathbf{Z}/2\mathbf{Z})$ is cyclic, we choose a generator t and set $\{f, g\} = i_*(t) \in H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$. We refer to the homology class $\{f, g\}$ as the *abelian cycle* corresponding to the pair f, g. In other words, an abelian cycle is just the pushforward of the fundamental class of $H_2(\mathbf{Z}^2)$. Abelian cycles in the second homology of a group also have a natural topological interpretation in terms of a torus embedded in a $K(G, 1)$-space.

Since the map i_* is not necessarily an injection, it is possible that the abelian cycle $\{f, g\}$ might be trivial. Our strategy will be to compute $\sigma_*(\{f, g\})$; if this is nonzero in $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$, then the original abelian cycle was also nontrivial.

We note that while we have given the definition of abelian cycles for the case of $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$, one can generalize the construction of abelian cycles in the obvious way for different choices of groups and of coefficients, and also for degree $n > 2$ by taking n commuting elements and looking at the injection $\mathbf{Z}^n \hookrightarrow \mathcal{K}$.

Basic homology calculations We recall that as a Z_2 linear vector space, B_2 is spanned by $d = \sum_{k=0}^{2} \begin{pmatrix} 2g \\ k \end{pmatrix} = 8g^2 + 2g + 1$ Boolean monomials. For the purposes of calculating $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$, it therefore suffices to think of $B_2 \cong \bigoplus_d Z_2$. Note that $\mathbf{RP}^\infty$ is a $K(\mathbf{Z}/2\mathbf{Z}, 1)$-space.

The Universal Coefficient Theorem gives us some insight into the structure of $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$ in the form of the following short exact sequence:

$$0 \to \wedge^2(B_2) \xrightarrow{\psi} H_2(B_2, \mathbf{Z}_2) \to \mathrm{Tor}(B_2, \mathbf{Z}_2) \to 0 \tag{3}$$

(See [Br] p. 126 (find Section Number) or [Hat], Section 3.A , e.g.)

A quick calculation shows that $\mathrm{Tor}(B_2, \mathbf{Z}/2\mathbf{Z}) \cong B_2$. Furthermore, the sequence splits, and we obtain

$$H_2(B_2, \mathbf{Z}/2\mathbf{Z}) \cong \wedge^2(B_2) \oplus B_2$$

We note that one could also use the Kunneth formula for homology groups of product space to express $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$ as a finite number of $\mathbf{Z}/2\mathbf{Z}$ summands, but this approach does not capture the useful structure of $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$ as revealed by the Universal Coefficient Theorem.

The following lemma is the key to proving Main Theorem 2, as it allows us to check that abelian cycles are nontrivial. It is an adaption of Lemma 5.3 of [Sa].

Lemma 3. *Let $\{f, g\}$ be an abelian cycle in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$. Then the following formula holds:*

$$\sigma_*(\{f, g\}) = (\sigma(f) \wedge \sigma(g), 0) \in \wedge^2 B_2 \oplus B_2$$

Proof. In terms of standard bar notation (see for example, [Br]), we have:

$$\sigma_*(\{f, g\}) = [\sigma(f)|\sigma(g)] - [\sigma(g)|\sigma(f)]$$

Further, it follows from the Universal Coefficient Theorem in the case of a finitely generated abelian group that the injection ψ in the short exact sequence (3) identifies the class of the cycle $[\sigma(f)|\sigma(g)] - [\sigma(g)|\sigma(f)]$ in $H_2(B_2, \mathbf{Z}/2\mathbf{Z}) \cong \wedge^2 B_2 \oplus B_2$ with $(\sigma(f) \wedge \sigma(g), 0)$. ⋄

Remark. As noted in the introduction, one could never hope to detect classes corresponding to the B_2 summand of $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$, since an abelian cycle arises from a map of $\mathbf{Z} \times \mathbf{Z}$ into $\mathcal{K}$ and any such class in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ must vanish in the Tor term of the Universal Coefficient Theorem. For this reason we will write $\wedge^2 B_2 \subset H_2(B_2, \mathbf{Z}/2\mathbf{Z})$ and will refer to the element $(x \wedge y, 0) \in H_2(B_2, \mathbf{Z}/2\mathbf{Z}) = \wedge^2 B_2 \oplus B_2$ simply as $x \wedge y \in \wedge^2 B_2$. Thus the formula of Lemma 3 becomes simply

$$\sigma_*(\{f, g\}) = \sigma(f) \wedge \sigma(g) \in H_2(B_2, \mathbf{Z}/2\mathbf{Z})$$

The formula of Lemma 3 will be our primary tool for proving the Main Theorem.

4 Proof of Main Theorem

Our strategy for proving Main Theorem 2 is twofold. First, we will use the induced map on homology σ_* together with Lemma 3 to hit basis elements of $\wedge^2 B_2$ directly with abelian cycles. Secondly, we will use the fact that σ is a $\mathrm{Mod}_{g,1}$-equivariant map to increase our efficiency. Thus we will compute orbits of basis elements of $\wedge^2 B_2$ and note that hitting one element of an orbit with σ_* tells us that every basis element in the orbit is also in the image of σ_*.

A basis for $\wedge^2 B_2$. Recall that a basis $\mathcal{B}$ for B_2 consists of $d = 8g^2+2g+1$ Boolean monomials. More precisely,

$$\mathcal{B} = \{1, \overline{a}_i, \overline{b}_i, \overline{a}_i\overline{b}_j, \overline{a}_i\overline{a}_j (i \neq j), \overline{b}_i\overline{b}_j (i \neq j)\}$$

where the a_i, b_j are the fixed symplectic basis for $H_1(S_{g,1}, \mathbf{Z}/2\mathbf{Z})$ shown in Figure 1. Thus a basis for $\wedge^2 B_2$ is given by

$$\begin{pmatrix} d \\ 2 \end{pmatrix} = 32g^4 + 16g^3 + 6g^2 + g$$

elements of the form $m_1 \wedge m_2$, where m_1, m_2 are distinct monomials in $\mathcal{B}$.

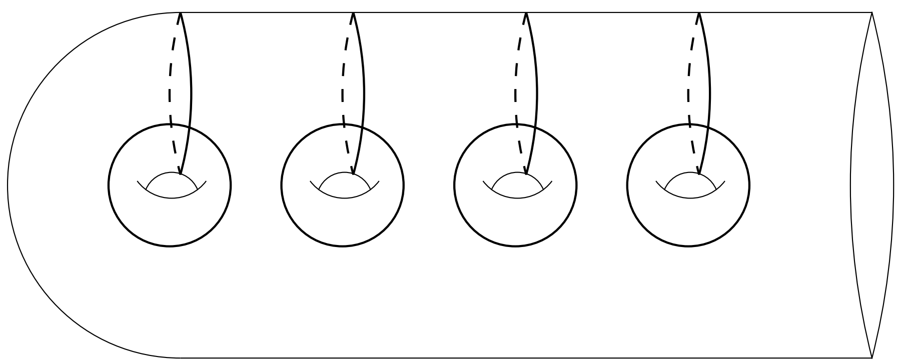

Figure 1: A fixed symplectic basis for$H_1(S_{g,1}, \mathbf{Z}/2\mathbf{Z})$ – BENSON, I haven't put in the labels yet, but the a_i are the meridians and the b_i are the longitudes.

When trying to describe a basis for $\wedge^2 B_2$, it is convenient to organize elements according to degree and also by distinct patterns of the indices occurring in the monomials involved. Also, as previously noted, it makes sense to partition our basis into $\mathrm{Mod}(S)$-orbits. For this purpose, we consider the action of the following two $\mathrm{Mod}(S)$-maps on the $\mathbf{Z}/2\mathbf{Z}$-vector space B_2:

$$\overline{a}_i \mapsto \overline{b}_i \mapsto \overline{a}_i \tag{4}$$

$$\overline{a}_i \mapsto \overline{a}_j; \quad \overline{b}_i \mapsto \overline{b}_j \tag{5}$$

Elements not mentioned in the above maps are understood to be fixed.

The list below contains all basis elements of $\wedge^2 B_2$. For the purposes of organization, we use Q (resp. L,C) to denote a quadratic (resp. linear, constant) monomial. We note that there is only one constant monomial, namely 1. Thus 'QQn' denotes those basis elements of the form (quadratic)$\wedge$(quadratic) which make up the n^{th} Mod(S)-orbit in our list, 'CLn' denotes the n^{th} orbit of basis elements of the form $1 \wedge (linear)$, etc. The ordering of orbits corresponds to the order in which we encounter these classes in the proof our Main Theorem, but within each orbit we give the elements in lexicographical order.
For all $i, j, k, l = 1, \ldots, g, \quad i \neq j \neq k \neq l$:

$$\begin{aligned}
\text{QQ1} &= \{\overline{a}_i\overline{b}_i \wedge \overline{a}_j\overline{b}_j\} \\
\text{QQ2} &= \{\overline{a}_i\overline{b}_i \wedge \overline{a}_j\overline{b}_k, \overline{a}_i\overline{b}_i \wedge \overline{a}_j\overline{a}_k, \overline{a}_i\overline{b}_i \wedge \overline{b}_j\overline{b}_k\} \\
\text{QQ3} &= \{\overline{a}_i\overline{a}_j \wedge \overline{a}_k\overline{a}_l, \overline{a}_i\overline{a}_j \wedge \overline{a}_k\overline{b}_l, \overline{a}_i\overline{a}_j \wedge \overline{b}_k\overline{b}_l, \overline{a}_i\overline{b}_j \wedge \overline{a}_k\overline{b}_l, \overline{a}_i\overline{b}_j \wedge \overline{a}_k\overline{b}_l, \overline{b}_i\overline{b}_j \wedge \overline{b}_k\overline{b}_l\} \\
\text{QQ4} &= \{\overline{a}_i\overline{a}_j \wedge \overline{a}_i\overline{a}_k, \overline{a}_i\overline{a}_j \wedge \overline{a}_i\overline{b}_k, \overline{a}_i\overline{b}_j \wedge \overline{a}_i\overline{b}_k, \overline{a}_i\overline{b}_j \wedge \overline{a}_k\overline{b}_j, \overline{a}_i\overline{b}_j \wedge \overline{b}_j\overline{b}_k, \overline{b}_i\overline{b}_j \wedge \overline{b}_i\overline{b}_k\} \\
\text{QQ5} &= \{\overline{a}_i\overline{b}_i \wedge \overline{a}_i\overline{b}_i\} \\
\text{QQ6} &= \{\overline{a}_i\overline{a}_j \wedge \overline{a}_i\overline{a}_j, \overline{a}_i\overline{b}_j \wedge \overline{a}_i\overline{b}_j, \overline{b}_i\overline{b}_j \wedge \overline{b}_i\overline{b}_j\} \\
\text{QQ7} &= \{\overline{a}_i\overline{b}_i \wedge \overline{a}_i\overline{b}_j, \overline{a}_i\overline{a}_j \wedge \overline{a}_i\overline{b}_i, \overline{a}_i\overline{b}_i \wedge \overline{b}_i\overline{b}_j, \overline{a}_i\overline{b}_i \wedge \overline{a}_j\overline{b}_i\} \\
\text{QQ8} &= \{\overline{a}_i\overline{a}_j \wedge \overline{a}_k\overline{b}_i, \overline{a}_i\overline{a}_j \wedge \overline{b}_i\overline{b}_k, \overline{a}_i\overline{b}_j \wedge \overline{a}_k\overline{b}_i, \overline{a}_i\overline{b}_j \wedge \overline{b}_i\overline{b}_k\} \\
\text{LQ1} &= \{\overline{a}_i \wedge \overline{a}_j\overline{b}_j, \overline{b}_i \wedge \overline{a}_j\overline{b}_j\} \\
\text{LQ2} &= \{\overline{a}_i \wedge \overline{a}_i\overline{b}_i, \overline{b}_i \wedge \overline{a}_i\overline{b}_i\} \\
\text{LQ3} &= \{\overline{a}_i \wedge \overline{a}_i\overline{a}_j, \overline{a}_i \wedge \overline{a}_i\overline{b}_j, \overline{b}_i \wedge \overline{a}_j\overline{b}_i, \overline{b}_i \wedge \overline{b}_i\overline{b}_j\} \\
\text{LQ4} &= \{\overline{a}_i \wedge \overline{a}_j\overline{a}_k, \overline{a}_i \wedge \overline{a}_j\overline{b}_k, \overline{a}_i \wedge \overline{b}_j\overline{b}_k, \overline{b}_i \wedge \overline{a}_j\overline{a}_k, \overline{b}_i \wedge \overline{a}_j\overline{b}_k, \overline{b}_i \wedge \overline{b}_j\overline{b}_k\} \\
\text{CQ1} &= \{1 \wedge \overline{a}_i\overline{b}_i\} \\
\text{CQ2} &= \{1 \wedge \overline{a}_i\overline{a}_j, 1 \wedge \overline{a}_i\overline{b}_j, 1 \wedge \overline{b}_i\overline{b}_j\} \\
\text{LL1} &= \{\overline{a}_i \wedge \overline{a}_i, \overline{b}_i \wedge \overline{b}_i\} \\
\text{LL2} &= \{\overline{a}_i \wedge \overline{a}_j, \overline{a}_i \wedge \overline{b}_j, \overline{b}_i \wedge \overline{b}_j\} \\
\text{LQ5} &= \{\overline{a}_i \wedge \overline{a}_j\overline{b}_i, \overline{a}_i \wedge \overline{b}_i\overline{b}_j, \overline{b}_i \wedge \overline{a}_i\overline{a}_j, \overline{b}_i \wedge \overline{a}_i\overline{b}_j\} \\
\text{LL3} &= \{\overline{a}_i \wedge \overline{b}_i\} \\
\text{CL1} &= \{1 \wedge \overline{a}_i, 1 \wedge \overline{b}_i\} \\
\text{CC1} &= \{1 \wedge 1\} \\
\text{QQ9} &= \{\overline{a}_i\overline{a}_j \wedge \overline{b}_i\overline{b}_j, \overline{a}_i\overline{b}_j \wedge \overline{a}_j\overline{b}_i\} \\
\text{QQ10} &= \{\overline{a}_i\overline{a}_j \wedge \overline{a}_i\overline{b}_j, \overline{a}_i\overline{b}_j \wedge \overline{b}_i\overline{b}_j\}
\end{aligned}$$

We are now able to give a precise statement of our Main Theorem.

Theorem 4. *The image of the map σ_* contains every orbit in the list of basis elements of $\wedge^2 B_2$ except QQ9 and QQ10. Further, the image of σ_* contains the following sums of elements from these classes (and their* Mod(S)*-orbits):*
Make a list here...

Each of the orbits QQ9 and QQ10 contains $g^2 - g$ elements. Since a basis for $\wedge^2 B_2$ has g^4 elements, the above theorem implies the lower bound on the rank of $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ given in Corollary 2 in the Introduction.

Direct calculations. We will work our way through the above-listed orbits one at a time, building on our previous calculations as we go. We shall give a couple of calculations explicitly in order to give the reader the idea of how to proceed. For the remaining cases, we shall give the readers the required abelian cycles and leave the calculations as an exercise.

Before we get started, we note a fact which simplifies our calculations. Let α, β be two simple closed curves on a surface such that $i(\alpha, \beta) = 1$. Then the boundary γ of a regular neighborhood N of $\alpha \cup \beta$ is a genus 1 separating curve. Further, the two curves α and β form a symplectic basis for $H_1(N, \mathbf{Z}/2\mathbf{Z})$, and hence $\sigma(T_\gamma) = \overline{\alpha}\overline{\beta}$ (in general we shall not distinguish between a curve and its homology class). We call the pair α, β a *spine* for the genus 1 separating curve γ. We shall also abuse terminology and refer to the spine of the map T_γ and the spine of the subsurface bounded by γ. Thus when trying to 'hit' a specific basis element of $\wedge^2 B_2$, we shall in general look first for spines which give the desired monomials. We also make the observation that disjoint spines correspond to disjoint separating curves and hence to commuting twists in $\mathcal{K}$.

Lemma 5. *The image of σ_* contains the orbit QQ1.*

Proof. Referring to Figure 2, let γ_i, γ_j be the two separating curves corresponding to the two 'spines' shown on the i^{th} and j^{th} holes, respectively. Then $\{T_{\gamma_i}, T_{\gamma_j}\}$ is an abelian cycle in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$, and we have:

$$\begin{aligned}\sigma_2(\{T_{\gamma_i}, T_{\gamma_j}\}) &= \sigma(T_{\gamma_i}) \wedge \sigma(T_{\gamma_j}) \\ &= \overline{a}_i\overline{b}_i \wedge \overline{a}_j\overline{b}_j\end{aligned}$$

□

Lemma 6. *The image of σ_* contains the orbit QQ2.*

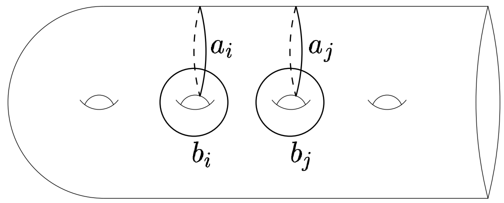

Figure 2: The spines of two bounding curves corresponding to an abelian cycle in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ which maps to an element of the orbit QQ1.

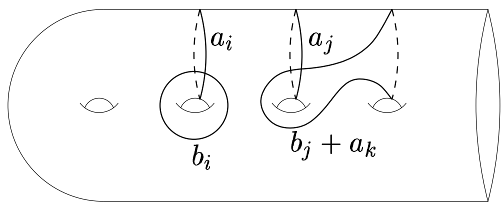

Figure 3: The spines of two bounding curves corresponding to an abelian cycle mapping to the orbit QQ2.

Proof. Let γ, δ denote the two separating curves corresponding to the spines shown in Figure 3; let the holes being 'used' in the picture be the i^{th}, j^{th}, and k^{th} holes, respectively. Then $\{T_\gamma, T_\delta\}$ is an abelian cycle, and

$$\begin{aligned}
\sigma_2(\{T_\gamma, T_\delta\}) &= \sigma(T_\gamma) \wedge \sigma(T_\delta) \\
&= \overline{a}_i \overline{b}_i \wedge \overline{a}_j \overline{c} \\
&= \overline{a}_i \overline{b}_i \wedge \overline{a}_j \overline{b_j + a_k} \\
&= \overline{a}_i \overline{b}_i \wedge \overline{a}_j (\overline{b}_j + \overline{a}_k + b_j \cdot a_k) \\
&= \overline{a}_i \overline{b}_i \wedge (\overline{a}_j \overline{b}_j + \overline{a}_j \overline{a}_k) \\
&= \overline{a}_i \overline{b}_i \wedge \overline{a}_j \overline{b}_j + \overline{a}_i \overline{b}_i \wedge \overline{a}_j \overline{a}_k
\end{aligned}$$

Now we are done since the first term on the right-hand side is already in the image of σ_2 by Lemma 5. In other words, we have found a sum of two abelian cycles which hit the orbit QQ2. □

SHOULD WE GIVE AN EXAMPLE INVOLVING A LINEAR AND/OR CONSTANT TERM?

Now we continue building in this way. Most of the following pictures show two spines corresponding to an abelian cycle in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$, except in the case where we use a genus 2 separating curve or when it is simply easier to draw the genus 1 separating curve itself. These cases will be noted.

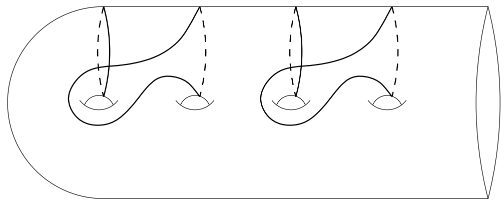

Figure 4: QQ3.

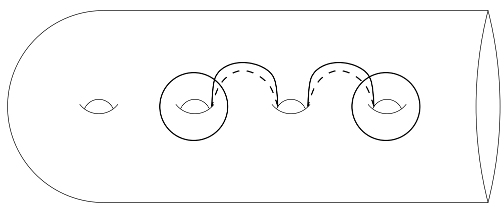

Figure 5: QQ4.

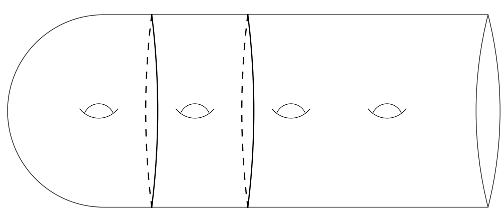

Figure 6: Two separating curves of genus 1 and genus 2, respectively, corresponding to the orbit QQ5.

In order to hit the orbit QQ6, we require the next two pictures taken together.

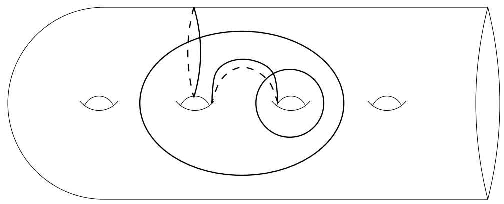

Figure 7: QQ6.

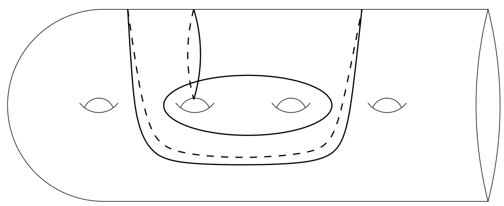

Figure 8: A spine and a genus 2 separating curve corresponding to an abelian cycle also needed for QQ6.

Continuing on with our list, beginning with the orbit QQ7:

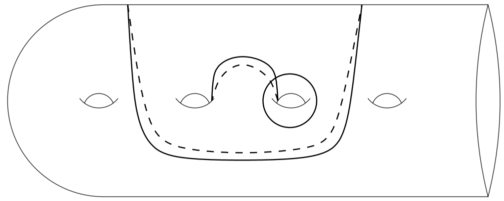

Figure 9: A spine and a genus 2 separating curve corresponding to orbit QQ7.

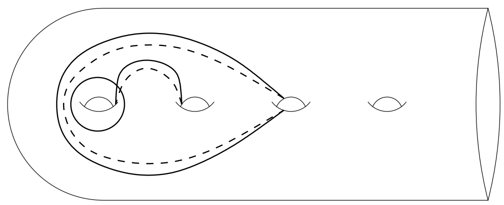

Figure 10: A spine and a genus 2 separating curve corresponding to orbit QQ8.

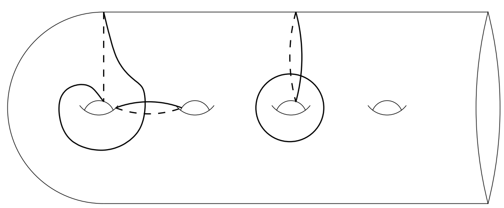

Figure 11: LQ1.

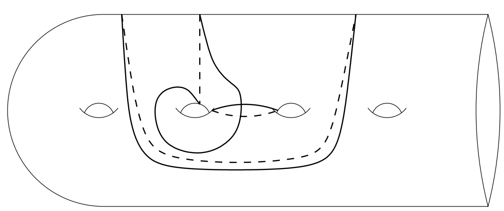

Figure 12: A spine and a genus 2 separating curve corresponding to orbit LQ2.

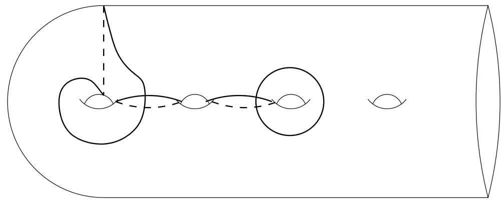

Figure 13: LQ3.

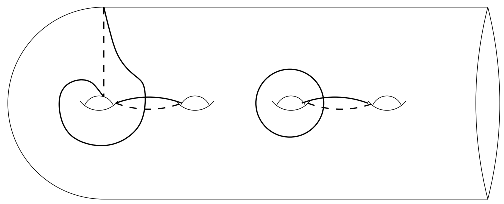

Figure 14: LQ4.

It turns out we can't get the last LQ-type term without hitting some other orbits first:

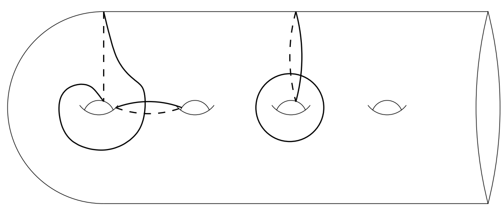

Figure 15: CQ1.

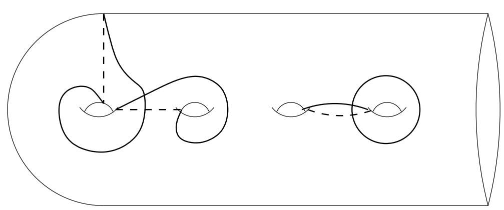

Figure 16: CQ2.

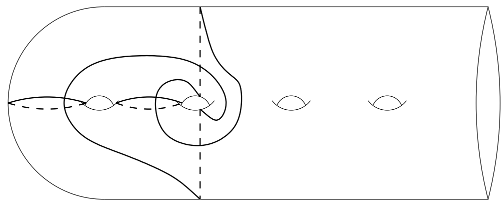

Figure 17: LL1.

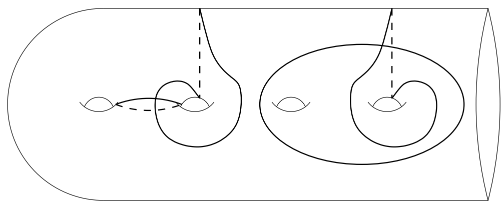

Figure 18: LL2.

Now we have enough to get LQ5:

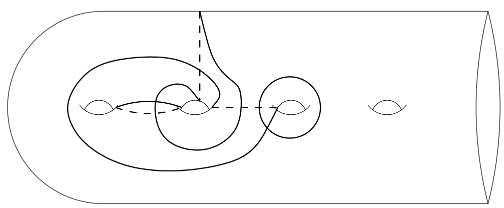

Figure 19: LQ5.

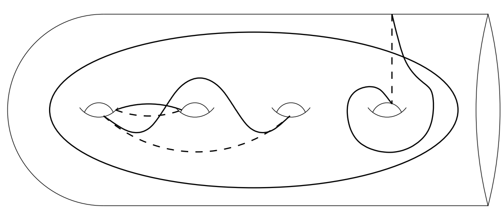

Figure 20: LL3.

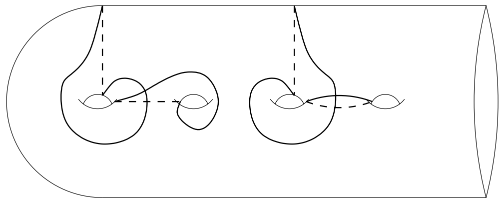

Figure 21: CL1.

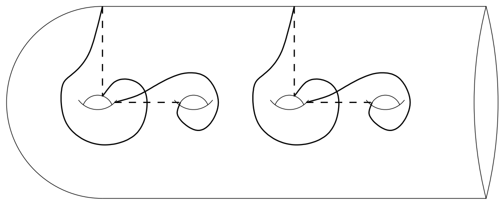

Figure 22: CC1.

Now, we can't get QQ9 and QQ10, but we can get certain sums of these basis elements, but I haven't drawn the pictures for this part yet.

And that's all for now...

5 Other stuff

D. Can hit B_3 too, use commutative diamond picture:

$$\mathcal{I} \to B_3 \to \wedge^3 H mod 2$$

$$\mathcal{I} \to \wedge^3 H \to \wedge^3 H mod 2$$

6 The Dual Picture

A. Give general cup, cap product stuff.

B. Start evaluating elements of $\sigma^* H^2 U$ on elements of $H_2(\mathcal{I})$

7 Questions

8 Appendix??

I'M NOT REALLY SURE WE NEED TO INCLUDE THIS...

The Kunneth formula for homology groups of product spaces with coefficients in an abelian group G gives us the following split exact sequence

$$0 \to \sum_{p+q=n} H_p(X) \otimes H_q(Y;G) \to H_n(X \times Y; G) \to \sum_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y;G)) \to 0$$

where $\mathbf{Z}$ coefficients are understood where coefficients are not otherwise specified. In fact, we get a natural isomorphism

$$\sum_{p+q=n} H_p(X;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H_q(Y;\mathbf{Z}_2) \cong H_2(X \times Y;\mathbf{Z}_2)$$

(Exercise 6.7 of p. 303 of Massey's Basic Course, taking $A = B = \emptyset$.) When $n = 2$, therefore, we get

$$H_2(X \times Y) \cong (H_0(X;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H_2(Y;\mathbf{Z}_2)) \oplus (H_1(X;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H_1(Y;\mathbf{Z}_2)) \oplus (H_2(X;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H_0(Y;\mathbf{Z}_2))$$

Now for all d we have

$$H_0(\underbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}_{d-1};\mathbf{Z}_2) = \mathbf{Z}_2$$

and

$$H_1(\underbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}_{d-1};\mathbf{Z}_2) = \bigoplus_{d-1} \mathbf{Z}_2$$

Then letting $X = \underbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}_{d-1}$ and $Y = \mathbf{R}P^\infty$ gives us

$$H_2(X \times Y;\mathbf{Z}_2) = (\mathbf{Z}_2) \oplus (\bigoplus_{d-1} \mathbf{Z}_2) \oplus (H_2(\underbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}_{d-1}) \otimes \mathbf{Z}_2$$

Thus starting with $H_2(\mathbf{R}P^\infty;\mathbf{Z}_2) = \mathbf{Z}_2$, we arrive inductively at

$$H_2(\bigoplus_d \mathbf{Z}_2;\mathbf{Z}_2) \cong \bigoplus_N \mathbf{Z}_2$$

where $N = \binom{d}{2} + d$.

Another version of the Kunneth formula which may be a bit simpler is as follows (all coefficients here are $\mathbf{Z}_2$, I am dropping them to keep the formula simpler).

$$H_2(\underbrace{\mathbf{R}P^\infty \times \cdots \times \mathbf{R}P^\infty}_{d}) = \bigoplus_{i_1+\cdots+i_n=2} (H_{i_1}(\mathbf{R}P^\infty) \otimes \cdots \otimes H_{i_n}(\mathbf{R}P^\infty))$$

This makes the value of N make more sense, assuming I haven't made any egregious errors in typing all this out.

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