

Research Statement

Mapping class groups of surfaces

Tara E. Brendle

My research is in the general area of geometric group theory, with a particular emphasis on mapping class groups of surfaces. The *mapping class group* $\text{Mod}(S)$ of a closed oriented surface S is the group of orientation-preserving homeomorphisms of S , up to isotopy, and is a fundamental object of study in both group theory and in low-dimensional topology. Mapping class groups are closely related to arithmetic groups and automorphism groups $\text{Aut}(F_n)$ of free groups, and are one generalization of braid groups. Thus the study of the algebraic structure of $\text{Mod}(S)$ has inspired more general group theoretic results, and also benefits from efforts to extend ideas about related groups to $\text{Mod}(S)$. Mapping class groups are also prominent in 3- and 4-manifold topology, e.g., via Heegaard splittings and Lefschetz fibrations. Moreover, $\text{Mod}(S)$ arises naturally as the fundamental group of the moduli space of Riemann surfaces, thus making the group an object of importance for both complex analysts and algebraic geometers (see [HL] for an excellent survey). Here are the general goals of my research program:

(1) Understand the algebraic structure of $\text{Mod}(S)$ and its various subgroups and thus understand the relationship between $\text{Mod}(S)$, arithmetic groups, and $\text{Aut}(F_n)$, by comparing properties such as linearity, actions on combinatorial models, automorphism and abstract commensurator groups, homology groups, generating sets, and finiteness properties.

(2) Clarify the relationship of $\text{Mod}(S)$ with 3-manifolds by studying subgroups of $\text{Mod}(S)$ which arise naturally from *Heegaard splittings*, that is, the result of gluing two handlebodies together along a common boundary via an element of $\text{Mod}(S)$ (every 3-manifold can be obtained in this way).

What follows is a more detailed discussion of several specific problems I work on, progress to date, and plans for the future.

Linearity. Mapping class groups share many of the same properties as finitely generated linear groups (for example, they satisfy a Tits alternative), but they also have much in common with the group of outer automorphisms of a free group, $\text{Out}(F_n)$. Formanek and Procesi showed that for $n \geq 3$, $\text{Aut}(F_n)$ is not linear, that is, it does not embed in $\text{GL}(n, k)$ for any field k ; it follows that $\text{Out}(F_n)$ is not linear when $n \geq 4$ [FP]. They did this by showing that a certain kind of HNN-extension which embeds in $\text{Aut}(F_n)$ is an obstruction to linearity. Lubotzky asked whether these obstructions exist in mapping class groups. Hamidi-Tehrani and I gave a generalization of the construction, which we call *FP-groups*. We answered Lubotzky's question in the negative [BHT]:

Theorem 1 (Brendle, Hamidi-Tehrani) *FP-groups do not embed in $\text{Mod}(S)$.*

The question of the linearity of $\text{Mod}(S)$ remains open, with no good candidates for a faithful representation, yet Theorem 1 says that the only available method for showing that a group is not linear which seemed likely to apply to $\text{Mod}(S)$ actually fails. For the moment, we are at an impasse, but I remain interested in the representation theory of $\text{Mod}(S)$.

Many of the problems I study involve the Torelli subgroup of $\text{Mod}(S)$, which I will now define. The mapping class group $\text{Mod}(S)$ acts naturally on $H = H_1(S, \mathbf{Z})$ while preserving the intersection form, giving rise to a surjective map known as the *symplectic representation*, $\rho : \text{Mod}(S) \rightarrow \text{Sp}(2g, \mathbf{Z}) \cong \text{Aut}(H)$ [MKS]. The kernel of the symplectic representation is called the *Torelli group* of the surface S , denoted $\mathcal{I} = \mathcal{I}(S)$. Since the map ρ and its image $\text{Sp}(2g, \mathbf{Z})$ are well understood, one approach to understanding $\text{Mod}(S)$ is to study the Torelli group. The Torelli group has become increasingly important in low dimensional topology, e.g., through Lefschetz fibrations and in the construction of homology 3-spheres via Heegaard splittings, as well as in algebraic geometry in work of Hain [Hai], among others.

In order to study \mathcal{I} , we introduce the elements of $\text{Mod}(S)$ known as Dehn twists. A *Dehn twist* about a simple closed curve c on S , denoted T_c , corresponds to cutting S along c , twisting a full 360 degrees to the right, and then regluing. In fact, $\text{Mod}(S)$ is generated by finitely many Dehn twists [De]. We now define a subgroup $\mathcal{K} < \mathcal{I}$ as follows:

$$\mathcal{K} = \{T_c \mid c \text{ is a separating curve on } S\}$$

The group \mathcal{K} , sometimes known as the *Johnson kernel*, is often used as a way to study \mathcal{I} , since it turns out that \mathcal{I}/\mathcal{K} is a finitely generated abelian group [Jo4]. Recently \mathcal{K} has emerged as interesting in its own right, especially in the study of homology 3-spheres. For example, while it is clear that splitting S^3 along a fixed Heegaard surface (i.e., an embedded surface which bounds a handlebody on both sides) and regluing via any element of \mathcal{I} yields a homology 3-sphere, Morita has shown that every homology 3-sphere may be obtained in this way using an element of \mathcal{K} [Mo1]. Further, Morita showed that the natural function from \mathcal{I} to \mathbf{Z} obtained by computing the Casson invariant of the homology 3-sphere obtained via this construction is in fact a homomorphism when restricted to \mathcal{K} (this is not true on \mathcal{I}).

It is known that \mathcal{I} is finitely generated (for genus ≥ 3) [Jo3] and that \mathcal{K} is not finitely generated [MM] [BiF]. It is not known whether \mathcal{I} is finitely presentable for $g \geq 3$. In fact, very little is understood about the algebraic structure of \mathcal{K} and \mathcal{I} .

Commensurations. Margalit and I studied the automorphism group of \mathcal{K} . Further, the *abstract commensurator* of a group Γ , denoted $\text{Comm}(\Gamma)$, is a generalization of the group of automorphisms of Γ . We define $\text{Comm}(\Gamma)$ as the group of isomorphisms of finite index subgroups of Γ (under composition), modulo the relation that two such isomorphisms are equivalent if they agree on a finite index subgroup of Γ . The product of $\phi_2\phi_1$, where $\phi_1 : G_1 \rightarrow G'_1$ and $\phi_2 : G_2 \rightarrow G'_2$, is defined on $\phi^{-1}(G'_1 \cap G'_2)$. We proved [BM]:

Theorem 2 (Brendle-Margalit) *For S a surface of genus at least 4, we have*

$$\text{Comm}(\mathcal{K}) \cong \text{Aut}(\mathcal{K}) \cong \text{Mod}(S)$$

Throughout the discussion of my work with Margalit, $\text{Mod}(S)$ will denote the group of orientation-reversing as well as orientation-preserving mapping classes. In general, the abstract commensurator of an arithmetic group is much larger than its automorphism group, whereas $\text{Comm}(\text{Out}(F_n)) \cong \text{Aut}(\text{Out}(F_n)) \cong F_n$ [FH], [BV]. In this respect, $\text{Mod}(S)$ and its subgroups \mathcal{I} and \mathcal{K} are more closely related to $\text{Out}(F_n)$ than arithmetic groups. We remark that Ivanov proved the analog of Theorem 2 for $\text{Mod}(S)$ [Iv]; Farb and Ivanov did the same for \mathcal{I} [FI]. Theorem 2 follows from:

Theorem 3 (Brendle-Margalit) *Let S be a surface of genus at least 4. If G is a finite index subgroup of the Johnson kernel \mathcal{K} , then any injection $\phi : G \rightarrow \mathcal{I}$ is induced by an element f of $\text{Mod}(S)$ in the sense that $\phi(h) = fhf^{-1}$ for all $h \in G$.*

We mention two corollaries of this work revealing basic algebraic structure of both \mathcal{I} and \mathcal{K} .

Corollary 4 (Brendle-Margalit) *If S is a surface of genus at least 4, then \mathcal{K} is co-Hopfian (as are all its finite index subgroups), that is, every injective endomorphism of \mathcal{K} is an isomorphism.*

Corollary 5 (Brendle-Margalit) *With the same hypotheses, \mathcal{K} is characteristic in \mathcal{I} .*

We note that the co-Hopfian property is shared by lattices in semisimple Lie groups (arithmetic groups in particular) [Pr], $\text{Out}(\mathbb{F}_n)$ (a result implicit in [BV] and proved explicitly for finite index subgroups of $\text{Out}(\mathbb{F}_n)$ in [FH]), and braid groups modulo their center [BeM].

Our methods, largely inspired by the work of Farb and Ivanov, actually recover their main results, which are the analogs of Theorems 2 and 3 in the case of the Torelli group:

Theorem 6 (Farb-Ivanov) *For a surface S of genus $g \geq 4$, we have:*

$$\text{Comm}(\mathcal{I}) \cong \text{Aut}(\mathcal{I}) \cong \text{Mod}(S)$$

Theorem 7 (Farb-Ivanov) *Under the same hypotheses, \mathcal{I} and all of its finite index subgroups are co-Hopfian. Further, for G a finite index subgroup of \mathcal{I} , any injection $\psi : G \rightarrow \mathcal{I}$ is induced by an element $f \in \text{Mod}(S)$ in the sense that $\psi(h) = fhf^{-1}$ for all $h \in G$.*

McCarthy and Vautaw have extended the second equality in Theorem 6 to the case $g \geq 3$ [MV].

Complexes of curves. Our strategy for proving Theorem 3 is to interpret algebraic statements about \mathcal{K} in terms of combinatorial topology. Complexes of curves on a surface can be viewed as analogs for $\text{Mod}(S)$ of buildings in the case of arithmetic groups, in the sense that they provide a combinatorial model for the group under consideration. The *curve complex* of a surface S , denoted $C(S)$, has a vertex for every simple closed curve on S , with an edge joining two vertices corresponding to disjoint curves. The *separating curve complex* $C_s(S)$ is the subcomplex of $C(S)$ spanned by vertices corresponding to separating curves on S . The *Torelli complex* $\mathcal{I}(S)$ is a slight modification of the Torelli geometry of Farb and Ivanov (their Torelli geometry $\mathcal{IG}(S)$ is $\mathcal{I}(S)$ plus a certain marking). The Torelli complex has a vertex corresponding to every separating curve and to every bounding pair on S . Edges join vertices which can be realized by disjoint curves.

We also look more generally at superinjective simplicial maps (introduced in [Ir]). A simplicial map ψ of $C(S)$, $C_s(S)$, or $\mathcal{I}(S)$ is *superinjective* if $i(\psi(a), \psi(b)) \neq 0$ whenever $i(a, b) = 0$ (here we mean the geometric intersection of the curves corresponding to vertices in the complex). We give results for $C_s(S)$ and $\mathcal{I}(S)$ analogous to two theorems of Ivanov [Iv] and of Irmak [Ir] for $C(S)$.

Theorem 8 (Brendle-Margalit) *For S a surface of genus at least 4, we have:*

$$\text{Aut}(\mathcal{I}(S)) \cong \text{Aut}(C_s(S)) \cong \text{Mod}(S)$$

Theorem 9 (Brendle-Margalit) *If S is a surface of genus at least 4, then every superinjective map of $C_s(S)$ is induced by an element of $\text{Mod}(S)$. Further, every superinjective map of $\mathcal{I}(S)$ is induced by an element of $\text{Mod}(S)$.*

These results also generalize similar results of Farb and Ivanov for $\mathcal{IG}(S)$ [FI]. We shall later discuss some possible applications of our methods to study the Heegaard subgroup of $\text{Mod}(S)$.

The Torelli generator problem. Johnson settled the fundamental question of finite generation for \mathcal{I} by showing that approximately 2^{2g} elements generate \mathcal{I} when the genus g of the surface S is at least 3 [Jo3]. On the other hand, his computation of an abelian quotient of \mathcal{I} gave a lower bound on the order of g^3 for the number of generators required for \mathcal{I} [Jo2]. Johnson achieved his lower bound in the case $g = 3$, [Jo3]. He conjectured that this bound was realizable in general [Jo6]:

In joint work with Benson Farb, we have an approach to proving Johnson's conjecture which has led to some surprising consequences along the way. Johnson's generators were carefully aligned with Humphries' minimal set of $(2g + 1)$ twist generators for $\text{Mod}(S)$. However, Dehn twists have infinite order in $\text{Mod}(S)$ which can present certain computational difficulties. Hence our first step was to seek new generating sets for $\text{Mod}(S)$ which would be computationally simpler. Luo recently described how to generate $\text{Mod}(S)$ with $12g + 6$ involutions (i.e., orientation-preserving elements of order 2) and posed the question of whether one could find a universal bound, independent of the genus g , for the number of torsion elements necessary to generate $\text{Mod}(S)$ [Lu]. Farb and I gave an affirmative answer to this question.

Theorem 10 (Brendle-Farb) *The mapping class group $\text{Mod}(S)$ is generated by three elements of finite order.*

Two of the three elements can be taken to be involutions, while the third has order linear in the genus of S . Korkmaz has since given a generating set for $\text{Mod}(S)$ consisting of two torsion elements, both of whose orders are linear in the genus of S [Ko1].

Farb and I also found a second generating set whose order as a set is still universally bounded, independent of genus, with the additional property that the order of each individual generator is also bounded, independent of genus.

Theorem 11 (Brendle-Farb) *The mapping class group $\text{Mod}(S)$ is generated by 6 involutions.*

We note that Kassabov has refined our methods to improve this to just 4 involutions when the genus of S is at least 8 and to 5 involutions when the genus is 6 or 7. Indeed, Theorems 10 and 11 have already inspired several papers (see, e.g., [Ko3], [Kot], [St], [Sz]). With many various generating sets for $\text{Mod}(S)$ now at our disposal, we have developed a few different approaches to proving this conjecture, including one which relies on a technique of factoring elements of the Torelli group (a complete set of relations in \mathcal{I} is not known) based on work in my Ph.D. thesis [Br]. This work is in progress at the time of this writing.

Connections with Coxeter groups. Coxeter groups are fundamental, well-understood objects of group theory, and relating them to $\text{Mod}(S)$ should bring new insights. We can apply Theorem 11 to give a new interpretation of $\text{Mod}(S)$ as a quotient of a Coxeter group. A *Coxeter group* is given by generators x_i , for $i = 1, \dots, n$ and defining relations $x_i^2 = 1$ for all $i = 1, \dots, n$ and $(x_i x_j)^{m_{i,j}} = 1$, where $m_{i,j} \in \{2, 3, 4, \dots, \infty\}$. If $m_{i,j} = \infty$, there is no relation between x_i and x_j . For example, the symmetric group is a Coxeter group with transpositions as generators. Theorem 11 above has the following implication, pointed out to us by Nathan Broaddus:

Corollary 12 (Brendle-Farb) *Mapping class groups are quotients of Coxeter groups.*

We note that one can obtain an *Artin group*, another generalization of a braid group, from a Coxeter presentation by removing the condition that $x_i^2 = 1$. Matsumoto and Labruère-Paris have

recently given an explicit presentation of $\text{Mod}(S)$ as the quotient of an Artin group. Broaddus and I are working on writing an analogous presentation of $\text{Mod}(S)$ as a quotient of a Coxeter group using the involution generators.

Homology of the Torelli group and the Johnson kernel. The homology of $\text{Mod}(S)$ is fairly well understood at least in low degree cases, thanks largely to work of Harer [Ha1], [Ha2]. In contrast, the only known (co)homology group of \mathcal{I} is its abelianization [Jo5]:

$$H_1(\mathcal{I}, \mathbf{Z}) \cong \wedge^3 H \oplus B_2$$

where $H \cong H_1(S, \mathbf{Z})$ and B_r denotes the $\mathbf{Z}/2\mathbf{Z}$ -vector space consisting of *Boolean* (or square-free) polynomials in $2g$ variables of degree at most r . No homology groups of \mathcal{K} are known. The calculation of the abelianization $H_1(\mathcal{K}, \mathbf{Z})$ would be of particular interest. Further, since the question of the finite presentation of \mathcal{I} remains open for genus at least 3, the group $H_2(\mathcal{I}, \mathbf{Z})$ is also important to know, since if it is not finitely generated, then this would imply that \mathcal{I} is not finitely presentable. The techniques used in this area in the past 15 years have come mainly from rational representation theory and algebraic geometry (see [Hai], [Sa], e.g.). In particular, these techniques yield little information about torsion and about the subgroup \mathcal{K} in \mathcal{I} .

In ongoing work, Benson Farb and I are studying the $\mathbf{Z}/2\mathbf{Z}$ -homology of \mathcal{K} and of \mathcal{I} . We use the *Birman-Craggs-Johnson homomorphism* $\sigma : \mathcal{I} \rightarrow B_3$ defined by Johnson [Jo2] using work of Birman and Craggs [BC], who defined a family of surjections $\mathcal{I} \rightarrow \mathbf{Z}/2\mathbf{Z}$ arising from the Rochlin invariant of spin 3-manifolds [BC]. The mapping class group $\text{Mod}(S)$ acts naturally on both \mathcal{I} and on B_3 , and σ is $\text{Mod}(S)$ -equivariant. Further, $\sigma(\mathcal{K}) = B_2$. We remark that if one tries to duplicate the algebraic-geometric techniques in this case, the modular representation theory questions which arise are beyond the reach of the current theory in that field.

As a starting point, we looked at the induced map on homology, restricted to \mathcal{K} in \mathcal{I} . Since B_2 as an abelian group is just a direct sum of copies of $\mathbf{Z}/2\mathbf{Z}$, it is easy to calculate its homology groups, and thus we have:

$$\sigma_* : H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z}) \rightarrow \wedge^2 B_2 \oplus B_2$$

We used the method of *abelian cycles* to give lower bounds on the rank of $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$. Roughly speaking, our main result says that we can hit most of the $\wedge^2 B_2$ summand of $H_2(B_2, \mathbf{Z}/2\mathbf{Z})$.

Theorem 13 (Brendle-Farb) *The image of σ_* contains all but two $\text{Mod}(S)$ -orbits of the summand $\wedge^2(B_2) \subset H_2(B_2, \mathbf{Z}/2\mathbf{Z})$.*

It follows from our construction that all homology classes in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ actually survive in $H_2(\mathcal{I}, \mathbf{Z}/2\mathbf{Z})$. We obtain the following corollaries:

Corollary 14 (Brendle-Farb) *The rank of $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$ is at least a polynomial in the genus g of S of degree 4.*

Corollary 15 (Brendle-Farb) *The rank of $H_2(\mathcal{I}, \mathbf{Z}/2\mathbf{Z})$ is at least a polynomial in the genus g of S of degree 4.*

Our method will work to give more classes in the full Torelli group, and we conjecture that we can improve our lower bound to a polynomial in g of degree 6 in this case. We are also working to understand the relationship between σ and the homology classes we have detected with it and previous work of Morita [Mo1] on a “lifting” of the BCJ homomorphism corresponding to the lift of the Rochlin invariant of 3-manifolds to the Casson invariant.

The Heegaard group. We now consider S as a fixed surface standardly embedded in S^3 . Then S is a *Heegaard surface*, that is, S bounds a handlebody on both sides. The *Heegaard subgroup* \mathcal{H} of the mapping class group $\text{Mod}(S)$ is the subgroup of $\text{Mod}(S)$ consisting of elements which extend to a homeomorphism of each of the two handlebodies. In 1933, Goeritz gave a finite set of generators for \mathcal{H} in the genus 2 case [Go]. Powell later proposed a certain finite set of elements of \mathcal{H} and gave a proof that they generate \mathcal{H} for higher genus [Pol]. However, his proof contains a gap, as recently discovered by Scharlemann [Sch]. Powell’s proposed generating set remains, in Scharlemann’s words, “a very plausible set [of generators]”. In ongoing joint work with Allen Hatcher, we are pursuing two approaches to this problem.

First, we want to understand the Hilden subgroup $\mathcal{A} < \mathcal{H}$, which is defined as the group of orientation-preserving homeomorphisms, up to isotopy, of the unit ball B in 3-space which leave invariant a set of g unknotted, unlinked arcs, properly embedded in B . We can see the inclusion of \mathcal{A} in \mathcal{H} by observing that $(B - \{\text{arcs}\})$ is a handlebody. Hilden gave a finite set of generators for \mathcal{A} [Hi], which Powell used as part of his proposed set of generators for \mathcal{A} . Hilden also described how \mathcal{A} embeds as a subgroup of the braid group on $2g$ strands. Hatcher and I are working on giving a finite presentation for \mathcal{A} .

The Hilden group \mathcal{A} can alternatively be thought of as the fundamental group of the space A of configurations of g disjoint smoothly and properly embedded arcs in upper-half space and which are unknotted and unlinked. We consider the space W of configurations of g disjoint semicircles properly embedded in upper-half space of \mathbf{R}^3 . Our goal is to prove a more general result, namely, that the inclusion map $W \hookrightarrow A$ is a homotopy equivalence. This statement implies that W is actually a $K(\pi, 1)$ -space for the Hilden group \mathcal{A} . Hilden observed two different embeddings of the braid group into \mathcal{A} which can be interpreted as “braiding” semicircles. The relations in our presentation for \mathcal{A} all arise as the obvious ones coming from this braiding.

Curve complexes, revisited. A second approach to the finite generation question for \mathcal{H} , which is also joint work with Dan Margalit, is inspired by Scharlemann’s new proof of Goeritz’s finite generation result for \mathcal{H} in genus 2 [Sch]. Scharlemann defines a complex of curves X , in which vertices correspond to simple closed curves on S which bound a disk on both sides of S in S^3 and edges join vertices v, w representing curves with geometric intersection equal to 4. Scharlemann’s complex is relatively easy to work with in the genus 2 case. However, there is no apparent reason why Scharlemann’s approach should not work in higher genus, but the situation will certainly be more complicated.

The key step in Margalit’s and my proof of Theorem 3 is to take a simplicial map of the separating curve complex $C_s(S)$ and obtain from it an automorphism of the entire curve complex $C(S)$. The key idea in this step is the notion of a *sharing pair*, that is, a pair of separating curves bounding genus 1 subsurfaces whose intersection is an annulus whose core is a nonseparating curve. A sharing pair of separating curves necessarily has geometric intersection number 4, and hence sharing pairs correspond to edges of Scharlemann’s complex if the curves involved bound a disk on either side of S . We developed a technique for studying sharing pairs using Harer’s *arc complex*, defined on a surface with boundary (originally defined in [Ha2]), as reinterpreted by Hatcher [Hat]. Such techniques, as well as those for the various curve complexes involved in the results outlined above will be of use in adapting Scharlemann’s proof to higher genus cases, or perhaps even in modifying Scharlemann’s complex appropriately in order to deal with the higher genus cases.

References

- [BeM] R. Bell & D. Margalit, *Braid groups are almost co-Hopfian*, arXiv:math.GT/0403145.
- [Bir] J. Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, **82**, Princeton University Press, 1974.
- [BC] J. Birman & R. Craggs, *The μ -invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold*, Trans. Amer. Math. Soc. **237** 1978, 283-309.
- [BB] J. Birman & T. Brendle, *Braids: a survey*, to appear as a chapter in “Handbook of Knot Theory”, ed. W. Menasco and M. Thistlethwaite; arXiv:math.GT/0409205.
- [BiF] D. Biss & B. Farb, *\mathcal{K} is not finite generated*, arXiv:math.GT/0405386.
- [Br] T. Brendle, *The Torelli group and representations of the mapping class group*, Doctoral Thesis, Columbia University, 2002 (<http://www.math.cornell.edu/~brendle/thesis.pdf>).
- [BF1] T. Brendle & B. Farb, *Every mapping class group is generated by 6 involutions*, J. Algebra, **278** 2004, 187-198.
- [BF2] T. Brendle & B. Farb, *The Birman-Craggs-Johnson homomorphism and abelian cycles in the Torelli group*, in preparation.
- [BHT] T. Brendle & H. Hamidi-Tehrani, *On the linearity problem for mapping class groups*, AGT, **1** 2001, 445-468.
- [BM] T. Brendle & D. Margalit, *Commensurations of the Johnson kernel*, Geometry and Topology, **8** 2004, 1361-1384.
- [BV] M. Bridson & K. Vogtmann, *Automorphisms of automorphism groups of free groups*, J. Algebra **229** (2) 2000, 785-792.
- [Bro] K. Brown, *Buildings*, Springer-Verlag, 1989.
- [De] M. Dehn, *The group of mapping classes*, Papers on group theory and topology, tr. J. Stillwell, Springer-Verlag, 1987 (*Die Gruppe der Abbildungsklassen*, Acta Math. **69** 1938, 135-206).
- [Fa] B. Farb, *Automorphisms of the Torelli group*, AMS Sectional Meeting, Ann Arbor, MI, March 1, 2002.
- [FH] B. Farb & M. Handel, *Commensurations of $\text{Out}(F_n)$* . Preprint, personal communication, September 2004.
- [FI] B. Farb & N. Ivanov, *The Torelli geometry and its applications* arXiv:math.GT/0311123.
- [FP] E. Formanek & C. Procesi, *The automorphism group of a free group is not linear*, J. Alg., **149** 1992, 494-499.
- [Go] L. Goeritz, *Die abbildungen der Brezelfläche und der Volbrezel vom Geschlecht 2*, Abh. Math. Sem. Univ. Hamburg **9** 1933, 244-259.

- [Hai] R. Hain, *Infinitesimal presentations of the Torelli groups*. J. Amer. Math. Soc. **10** (3) 1997, 597–651.
- [HL] R. Hain & E. Looijenga, *Mapping class groups and moduli spaces of curves*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., AMS **62** (2) 1997, 97-142.
- [Ha1] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Inv. Math. **72** 1982, 221-239.
- [Ha2] J. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Annals of Math. **121** 1985, 215-249.
- [Hat] A. Hatcher, *On triangulations of surfaces*, Top. Appl. **40** (2) 1991, 189-194.
- [Hi] H. Hilden, *Generators for two groups related to the braid group*, Pac. J. Math. **59** (2) 1975, 475-486.
- [Hu] S. Humphries, *Generators for the mapping class group*, Topology of Low-Dimensional Manifolds, Proc. Second Sussex Conf., Lecture Notes in Math., **722** Springer-Verlag 1979, 44-47.
- [Ir] E. Irmak, *Superinjective simplicial maps of complexes of curves and injective homomorphisms of subgroups of mapping class groups*, Topology, **43**(3) 2004, 513-541.
- [Iv] N. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces*, Internat. Math. Res. Notices **14** 1997, 651-666.
- [Jo1] D. Johnson, *An abelian quotient of the mapping class group \mathcal{I}_g* , Math. Ann. **249** (3) 1980, 225-242.
- [Jo2] D. Johnson, *Quadratic forms and the Birman-Craggs homomorphisms*, Trans. Amer. Math. Soc. **261** (1) 1980, 235-254.
- [Jo3] D. Johnson, *The structure of the Torelli group. I. A finite set of generators for \mathcal{I}* , Ann. of Math. (2) **118** (3) 1983, 423-442.
- [Jo4] D. Johnson, *The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves*, Topology **24** (2) 1985, 113-126.
- [Jo5] D. Johnson, *The structure of the Torelli group. III. The abelianization of \mathcal{T}* , Topology **24** (2) 1985, 127-144.
- [Jo6] D. Johnson, *A survey of the Torelli group*, Contemp. Math., AMS, **20** 1983, 165-179.
- [Ko1] M. Korkmaz, *Generating the surface mapping class group by two elements*, arXiv:math.GT/0309299.
- [Ko2] M. Korkmaz, *Low-dimensional homology groups of mapping class groups: a survey*. Mustafa Korkmaz, Proc. of 8th Gokova Geometry-Topology Conference, Turkish Journal of Mathematics **26** (1) 2002, 101-114.
- [Ko3] M. Korkmaz, *On a question of Brendle and Farb*, arXiv:math.GT/0307146.

- [Kot] D. Kotschick, *Quasi-homomorphisms and stable lengths in mapping class groups*, Proc. Amer. Math. Soc. **132** 2004, 3167-3175.
- [LP] C. Labruère & L. Paris, *Presentations for the punctured mapping class groups in terms of Artin groups*, Algebra. Geom. Topol. **1** 2001, 73-114.
- [Le] C. Leininger, *On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number* **8** 2004, 1301-1359.
- [Lu] F. Luo *Torsion elements in the mapping class group of a surface*, arXiv:math.GT/0004048.
- [MKS] W. Magnus. A. Karrass & D. Solitar. *Combinatorial group theory*, Dover Publications, Inc., 1976.
- [Ma] M. Matsumoto *A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities*, Math. Ann. **316** 2000, 401-418.
- [MV] J. McCarthy & W. Vautaw, *Automorphisms of Torelli groups*, arXiv:math.GT/0311250.
- [Me] G. Mess, *The Torelli groups for genus 2 and 3 surfaces*, Topology **31** (4) 1992, 775-790.
- [MM] D. McCullough & A. Miller, *The genus 2 Torelli group is not finitely generated*, Topology Appl. **22** (1) 1986, 43-49.
- [Mo1] S. Morita, *Casson's invariant for homology 3-spheres and characteristic classes of surface bundles I*, Topology **28** (3) 1989, 305-323.
- [Mo2] S. Morita, *Structure of the mapping class groups of surfaces: a survey and a prospect*, Proc. of the Kirbyfest, GT Monograph **2** 1999, 349-406.
- [Po1] J. Powell, *Homeomorphisms of S^3 leaving a Heegaard surface invariant*, Trans. Amer. Math. Soc. **257** (1) 1980, 193-216.
- [Pr] G. Prasad, *Discrete subgroups isomorphic to lattices in semisimple Lie Groups*, Amer. J. Math. **98** (1) 1976, 241-261.
- [Sa] T. Sakasai, *The Johnson homomorphism and the third rational cohomology group of the Torelli group*, UTMS preprint **2003-21**, (<http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2003-21.pdf>).
- [Sch] M. Scharlemann, *Automorphisms of the 3-sphere preserving a genus 2 Heegaard splitting*, arXiv:math.GT/0307231.
- [St] M. Stukow, *The extended mapping class group is generated by 3 symmetries*, C. R. Math. Acad. Sci. Paris **338** (5) (2004), 403-406.
- [Su] M. Suzuki, *Irreducible decomposition of the Magnus representation of the Torelli group*, Bull. Austral. Math. Soc. **67** 2003, 1-14.
- [Sz] B. Szepietowski, *Mapping class group of non-orientable surface is generated by three elements and by four involutions*, preprint (<http://math.univ.gda.pl/blaszep/papers/4inv01.ps>).
- [Wa] B. Wajnryb, *Mapping class group of a surface is generated by two elements*, Topology **35** (2) 1996, 377-383.