Summary of Papers

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Completed papers.

- *The Torelli group and representations of the mapping class group,* Doctoral Thesis, Columbia University, May 2002 (http://www.math.cornell.edu/ brendle/thesis.pdf).

  We generalize a method of Dennis Johnson to find relations amongst his finite generating set for the Torelli group $\mathcal{I}$ of a surface S and give an alternate technique which yields a family of relations in $\mathcal{I}$ which can be expressed as commutators in $\mathrm{Mod}(S)$. We also make explicit the relationship of the symplectic representation of $\mathrm{Mod}(S)$, which measures the action of $\mathrm{Mod}(S)$ on homology and of which the Torelli group is the kernel, to two linear representations of $\mathrm{Mod}(S)$ due to Trapp (and independently Sipe) and to Perron. Both representations are essentially linearizations of a map of Morita extending Johnson's homomorphism $\tau : \mathcal{I} \to \wedge^3(H_1(S, \mathbf{Z}))$ and encoding the action of $\mathrm{Mod}(S)$ on winding numbers of curves on S as well as symplectic information. This thesis also contains joint work with Hessam Hamidi-Tehrani, cited below.

- *On the linearity problem for mapping class groups,* (with Hessam Hamidi-Tehrani), Algebraic and Geometric Topology, **1**, 2001, 449-468.

  It is an open problem to determine if the mapping class group $\mathrm{Mod}(S)$ is linear for genus $g \geq 3$. We show that certain obstructions to linearity which were discovered by Formanek and Procesi in $\mathrm{Aut}(F_n)$ for $n \geq 3$ do not exist in $\mathrm{Mod}(S)$. This answers a question of Lubotzky, who suggested that Formanek and Procesi's methods could perhaps be used to show that $\mathrm{Mod}(S)$ is not linear. In particular, our result eliminates as a possibility what was essentially the only available approach to proving that $\mathrm{Mod}(S)$ is not linear. This work also appeared as part of the first author's Ph.D. thesis, cited above.

- *Every mapping class group is generated by 6 involutions,* (with Benson Farb), Journal of Algebra, Vol. 278 (2004), 187-198.

  We answer a question of Feng Luo by giving a bound independent of genus for the number of involutions necessary to generate $\mathrm{Mod}(S)$, the mapping class group of a surface S of genus g at least 3. We also give a generating set consisting of 3 finite order elements (2 of which can be taken to be involutions, the third of which has order linear in g). This work was motivated by an approach of the authors (in ongoing work) have to find new generating sets for the Torelli subgroup of $\mathrm{Mod}(S)$ in which it is convenient for many calculations to have such a generating set for $\mathrm{Mod}(S)$.

- *Commensurations of the Johnson kernel,* (with Dan Margalit), Geometry and Topology, Vol. 8 (2004), 1361-1384.

  Let $\mathcal{K}$ denote the *Johnson kernel* $\mathcal{K}$, that is, the subgroup of the Torelli group generated by Dehn twists about separating curves on S.

  We prove that when the genus of S is at least 5, $\mathrm{Comm}(\mathcal{K}) \cong \mathrm{Aut}(\mathcal{K}) \cong \mathrm{Mod}(S)$. In particular, this verifies a conjecture of Farb. Here $\mathrm{Comm}(\mathcal{K})$ denotes the abstract commensurator of $\mathcal{K}$, and $\mathrm{Mod}(S)$ denotes the *extended mapping class group* of S, that is, the group of both orientation-preserving and orientation-reversing mapping classes of S. This result follows from our theorem: every injection ϕ of $\mathcal{K}$ into $\mathcal{I}$ is induced by an element of $\mathrm{Mod}(S)$ in the sense that $\phi(h) = fhf^{-1}$ for some $f \in \mathrm{Mod}(S)$. As corollaries we prove that $\mathcal{K}$ and all its finite index subgroups have the co-Hopfian property (every injective endomorphism is an isomorphism) and that $\mathcal{K}$ is characteristic in the Torelli group. Further, our methods recover analogous theorems of Farb and Ivanov in the case of the Torelli group.

  Our method is to translate algebraic data into combinatorial topology. For example, we prove that the the automorphism group of the complex of separating curves on S is isomorphic to $\mathrm{Mod}(S)$. We note that Ivanov has shown that the automorphisms of the full curve complex are isomorphic to $\mathrm{Mod}(S)$; our results show that an action on separating curves determines an action on nonseparating curves.

- *Braids: a survey,* (with Joan S. Birman), to appear in the "Handbook of Knot Theory", ed. William W. Menasco and Morwen B. Thistlethwaite, arXiv:math.GT/0409205.

  We give a survey of the role of the braid group in knot theory which updates Birman's book "Braids, links and mapping class groups". The aim of the article is both to provide a starting point for graduate students hoping to work in the area as well as to give an overview for experts of recent developments in the field. For example, we give simplified versions of proofs of classical results such as Alexander's Theorem and Markov's Theorem due to Yamada (with improvements by Vogel) and by Traczyk, respectively, which are more suitable for computer programming than other known proofs. A common theme is the relationship of braid groups to their various generalizations, such as surface mapping class groups, Artin groups, and Garside groups. We plan to expand the survey article into a book.

Papers in preparation.

1. *The Birman-Craggs-Johnson homomorphism and abelian cycles in the Torelli group,* (with Benson Farb), (in preparation).

   We construct the first examples (to the best of our knowledge) of nontrivial homology classes in $H_2(\mathcal{K}, \mathbf{Z}/2\mathbf{Z})$, which also give nontrivial classes in $H_2(\mathcal{I}, \mathbf{Z}/2\mathbf{Z})$. In particular, we prove that the rank of each of these groups is at least a polynomial of degree 4 in the genus g. Our methods are to construct abelian cycles in these homology groups and then to use a map of $\mathcal{I}$ arising from the Rochlin invariant of homology spheres, the Birman-Craggs-Johnson homomorphism $\sigma : \mathcal{I} \to B_3$, where B_3 is a $\mathbf{Z}/2\mathbf{Z}$-vector space and a subset of the algebra of Boolean polynomials, to check that these classes are non-vanishing.

In ongoing work, we are trying to improve this lower bound in the case of $\mathcal{I}$ as well as to classify all such abelian cycles. While in some sense this method lifts integrally using Morita's map to the Casson-Morita algebra $\mathcal{A}$, we are able to construct more classes explicitly when working modulo 2. One novelty here is that questions in modular representation theory which arise here seem to be beyond the reach of current knowledge in that area, so we take a more geometric approach.

2. *Wicket subgroups of braid groups* (with Allen Hatcher) In work in progress, we seek a finite presentation for the Hilden subgroup of the mapping class group $\mathrm{Mod}(S)$, which is isomorphic to the group of orientation-preserving homeomorphisms (up to isotopy) of the unit ball B in 3-space which leaves invariant a set of g arcs, properly embedded in B. It is equivalent to define the Hilden group $\mathcal{A}$ as the fundamental group of the space A of configurations of g disjoint smoothly and properly embedded arcs in upper-half space and which are unknotted and unlinked. Hilden has given a set of generators of $\mathcal{A}$, which have interpretations as "braiding" amongst the different arcs. We plan to show that the only relations amongst these generators are the obvious relations that arise from this braiding of arcs.

   Our method is to consider the space W of configurations of g disjoint semicircles properly embedded in upper-half space. Our goal is to show that the inclusion map $W \hookrightarrow A$ is a homotopy equivalence. It then follows relatively easily that W is actually a $K(\pi, 1)$-space for the Hilden group $\mathcal{A}$.