A REMARK ON A MAXIMUM PRINCIPLE

BO YANG AND FANGYANG ZHENG

ABSTRACT. We review the maximum principle in our CAG 2013 paper, and correct some inaccuracies in the proof.

Theorem 0.1. Let \( u(x,t) \) be a smooth solution of \( \frac{\partial u}{\partial t} = \Delta u + |u|^p \) with \( p > 1 \) on \( \mathbb{R}^n \times [0,T] \) with \( u(x,0) \geq 0 \). Then \( u(x,t) \geq 0 \) for any \( (x,t) \in \mathbb{R}^n \times [0,T] \).

The proof of above relies on the choice of the following cut-off function. For fixed \( p > 1 \), \( \varphi \) is a fixed smooth cut-off non-increasing function such that \( \varphi = 1 \) on \((-\infty,1)\) and \( \varphi = 0 \) on \([2,+\infty)\). and there exists \( C > 0 \),

\[
-C < \varphi' \leq 0, \quad \frac{|\varphi'|}{\varphi^{\frac{1}{p}}} + \frac{|\varphi''|}{\varphi^{\frac{3}{p}}} \leq C.
\]

Theorem 0.2 (Yang-Zheng). Let \( g(t) \) be a complete solution of the Kähler-Ricci flow on \( \mathbb{C}^n \) with \( U(n) \)-symmetry for \( t \in [0,T] \). If the Riemannian sectional curvature of the initial metric \( g(0) \) is nonnegative, so is that of \( g(t) \) for any \( t \in (0,T) \).

Proof explained. Note that one can assume \( A,B,C > 0 \) everywhere on \( \mathbb{C}^n \times [0,T] \). Suppose there is a point \((z_0,t_0)\) where \( 0 < t_0 \leq T \) where the sectional curvature is negative along some real 2-plane, then \( D(z_0,t_0) = AC - B^2 < 0 \). By picking \( r_0 > 0 \) small enough we may assume that \( \text{Ric}(z,t) \leq \frac{n-1}{r_0^2} \) for any \( z \in B_{r_0}(z_0,r_0) \) where \( B_{r_0}(z_0,r_0) \) is with respect to \( g(t_0) \).

\[
\frac{\partial}{\partial t} - \Delta(Ax - B^2) = \left[ \left( \frac{\partial}{\partial t} - \Delta \right) A \right] x + \left[ \left( \frac{\partial}{\partial t} - \Delta \right) C \right] x - 2B \left[ \left( \frac{\partial}{\partial t} - \Delta \right) B \right] - 2Ax \cdot \nabla x + 2|\nabla x|^2.
\]

Let \( \varphi \) is a fixed smooth cut-off non-increasing function such that \( \varphi = 1 \) on \((-\infty,1)\) and \( \varphi = 0 \) on \([2,+\infty)\). Moreover,

\[
-4 < \varphi' \leq 0, \quad \frac{|\varphi''|}{\varphi^{\frac{1}{p}}} + \frac{(\varphi')^2}{\varphi^{\frac{3}{p}}} \leq 128.
\]

Define \( u(z,t) \equiv \varphi \left( \frac{d(z,z_0)}{ar_0} \right) D(z,t), \) where \( a > 0 \) will be a sufficiently large number.

\[
\frac{\partial}{\partial t} - \Delta u = \varphi' \left( \frac{1}{ar_0} \right) \left[ \left( \frac{\partial}{\partial t} - \Delta \right) D \right] + \varphi \left[ \left( \frac{\partial}{\partial t} - \Delta \right) D \right] - 2A \varphi' \cdot \nabla D - \frac{D}{(ar_0)^2} D
\]

Denote \( u_{\text{min}}(t) = \min_{z \in \mathbb{C}^n} u(z,t) \), so \( u_{\text{min}}(t_0) \leq u(z_0,t_0) < 0 \). Assume that there exists \((z_1,t_1)\) such that \( u(z_1,t_1) = \min_{z \in \mathbb{C}^n} u(z,t) < 0 \). Now we compute the right hand side of (4) at the space-time point \((z_1,t_1)\). For simplicity, let us call it \( Q(z_1,t_1) \).

First of all, Lemma 8.3 from Perelman implies:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) d_{t_0}(z,z_0) \geq -\frac{5(n-1)}{3r_0},
\]

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whenever \( d_t(z, z_0) > r_0 \).

The definition of \((z_1, t_1)\) implies \( \nabla u(z_1, t_1) = 0 \). Therefore \( \nabla D = -\frac{\nabla \varphi}{\varphi} D \) and \( \nabla A = \frac{1}{C}(\nabla D + 2B \nabla C) \).

It follows from the \( F(x) \) function characterization of \( U(n) \)-invariant Kähler metric and a straightforward calculation that

\( \nabla A B = \frac{2x}{v}(A - 2B), \quad \nabla A C = \frac{2x}{v}(2B - C) \).

(6)

\[ Q(x_1, t_1) \geq \varphi \left[ A^2 C + (n - 2) B^2 C + \frac{n}{2} C^2 A + 2 B^3 - 2 \nabla A \cdot \nabla C + 2 |\nabla B|^2 \right] \]

\[ \quad - \frac{10(n - 1) \varphi'}{3 ar_0^2} D + \frac{2(\varphi')^2}{(ar_0)^2} D - \frac{\varphi'' D}{(ar_0)^2} \]

\[ = \varphi \left[ A^2 C + (n - 2) B^2 C + \frac{n}{2} C^2 A + 2 B^3 \right] \]

\[ + \varphi \left[ - \frac{2}{C} \nabla D \cdot \nabla C - \frac{4B}{C} \nabla B \cdot \nabla C + \frac{2A}{C} |\nabla C|^2 + 2 |\nabla B|^2 \right] \]

\[ - \frac{10(n - 1) \varphi'}{3 ar_0^2} D + \frac{2(\varphi')^2}{(ar_0)^2} D - \frac{\varphi'' D}{(ar_0)^2} \]

\[ \geq \varphi \left[ A^2 C + (n - 2) B^2 C + \frac{n}{2} C^2 A + 2 B^3 \right] \]

\[ + \varphi \frac{4x^2}{C v^2} \left[ A^2 C + AC^2 + 8 B^3 - 6ABC \right] \]

\[ - \varphi \frac{1}{ar_0 C v} |B - C| D - \frac{10(n - 1) \varphi'}{3 ar_0^2} D + \frac{2(\varphi')^2}{(ar_0)^2} D - \frac{\varphi'' D}{(ar_0)^2} \]

Note that at the point \((z_1, t_1)\),

(8)

\[ A^2 C + (n - 2) B^2 C + \frac{n}{2} C^2 A + 2 B^3 \geq |B^2 - AC|^\frac{1}{2} = |D|^\frac{1}{2}, \]

(9)

\[ A^2 C + AC^2 + 8 B^3 - 6ABC \geq 0. \]

Claim 0.3. \( \frac{2|B - C|}{Cv} \) is uniformly bounded on \( \mathbb{C}^n \times [0, T] \).

Proof of Claim. The crucial observation is that \( x - \frac{4}{v} = O(x^3) \) and \( \frac{2x}{v} - 1 - \frac{1}{\sqrt{1 + F(x)}} = O(x^3) \) when \( x \) small, which gives \( \frac{2|B - C|}{Cv} \) is bounded when \( x \) small.

Indeed, \( v = x^2 + \frac{|F''(x)|^2}{4} x^4 + O(x^5) \) and \( \sqrt{1 + (F')^2} = 1 + \frac{|F''(x)|^2}{4} x^2 + O(x^3) \), then one can check that \( 2 \sqrt{1 + (F')^2} - \frac{1}{v} \sqrt{1 + (F')^2} x^2 = O(x^3) \) and \( 1 - \frac{x^2}{v} = \frac{(F'(x))^2}{4} x^2 + O(x^3) \).

On the other hand, \( C \geq \frac{4}{v} \) for \( x \) large leads to \( \frac{2|B - C|}{Cv} \) is bounded outside a compact set of \( \mathbb{C}^n \). In fact,

(10)

\[ \lim_{x \to +\infty} x \frac{|2B - C|}{Cv} = 0. \]

It follows from (7) that

\[ \frac{d}{dt} u_{\min}(t) \big|_{t=t_1} \geq \frac{1}{\varphi} \left[ |u|^\frac{3}{2} + \left[ - \frac{\varphi'}{ar_0 \varphi^2} C_1 + \frac{\varphi'}{ar_0^2 \varphi^2} C_2 + \frac{(\varphi')^2 C_3}{(ar_0)^2 \varphi^2} + \frac{|\varphi''|}{(ar_0)^2 \varphi^2} |u| \right] \right] \]

where \( C_1, C_2 \) and \( C_3 \) are all constants depending only on the \( g(t) \) restricted to a compact subset \( \mathbb{C}^n \times [0, T] \).
On the other hand, the choice of the point \((z_1, x_1)\) implies \(\frac{d^n}{dt^n} u_{\text{min}}(t) \leq 0\). We conclude that 
\[
\sqrt{|u(x_1, t_1)|} \leq \frac{C_5}{ar_0} + \frac{C_6}{(ar_0)^2}.
\]
Therefore, we have
\[
\frac{D(x_0, t_0) \geq u(x_1, t_1) \geq -\left[\frac{C_5}{ar_0} + \frac{C_6}{(ar_0)^2}\right]^2.}
\]

Now let \(a\) goes to infinity, we get \(D(z_0, t_0) \geq 0\), which contradicts to the choice of \((z_0, t_0)\). \(\square\)