A SHORT NOTE ON A LEMMA OF CHEN-SUN-TIAN

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A lemma from Chen-Sun-Tian [2] states that the normalized Kähler-Ricci flow on compact Kähler manifolds with positive holomorphic bisectional curvature remains to have positive holomorphic bisectional curvature bounded from below assuming that Ricci curvature has a positive lower bound along the flow. We will show that the lemma also follows from the general result of Böhm-Wilking [1].

1. NOTATIONS

We follow Ni-Wallach [3] for the computation in the Kähler analogue of Böhm-Wilking [1]. Let \((M^m, g)\) be a Kähler manifold, \(\omega\) its Kähler form and \(\Omega\) the space of real \((1,1)\) form, then curvature operator of \(M^m\) can be viewed as a symmetric transformation on \(\Omega\) by

\[
R(\sqrt{-1}\Omega_{ij} \omega^i \wedge \omega^j) = \sqrt{-1} R_{ijkl} \Omega_{ij} \omega^k \wedge \omega^l
\]

where we use unitary frame \(\{\omega^i\}\) \(1 \leq i \leq m\).

Following Ni-Wallach, we further define a general product \(A \wedge B\) for \((1,1)\) forms by

\[
(A \wedge B)_{ijkl} = A_{ij} B_{kl} + A_{kl} B_{ij} + A_{ik} B_{lj} + A_{ik} B_{lj}
\]

Then one can write the following decomposition for Kähler curvature operator:

\[
R = \frac{S}{2m(m+1)} \omega \wedge \omega + \frac{1}{m+2} (\text{Ric} - \frac{1}{m+2} \omega) \wedge \omega + W
\]

Define \(\text{Ric}(R)\) to be \(\text{Ric}(R)_{ij} = R_{ijkl} \Omega_{ij} \Omega_{kl}\), then \(W\) in the above satisfies \(\text{Ric}(W) = 0\). Denote the above decomposition by \(\tilde{R} = R_I + R_{\text{Ric}} + R_W\), one can also define inner product on the space of algebraic Kähler curvature operator to make the above decomposition be orthogonal.

Further define

\[
Rc^0 = \text{Ric} - \frac{S}{m} \omega
\]

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\(^1\)This short note was written around June 2009 after I reported Chen-Sun-Tian [2] at the informal differential geometry seminar at UCSD organized by Ben Chow, Lei Ni, and Ben Weinkove. This lemma is proved in [2] by applying tensor maximum principle on compact manifolds and checking the null vector condition. Lei Ni suggested that the general argument in Böhm-Wilking [1] might also be used to show the lemma. It turns out to be an exercise to follow [1] to do so. Note that what we did here is no more than a reformulation of the maximum principle argument in [2]. Although the convergence of such a flow is clear after the solution of Frankel conjecture and the work of Chen-Tian. It is still interesting to understand curvature pinching along Kähler-Ricci flow on compact Kähler manifolds with positive bisectional curvature, and it could have applications on Kähler-Ricci flow on complete manifolds.
\[ I = \frac{1}{2} \omega \bar{\omega} \]

\[(R^2)_{ijkl} = R_{ijpq} R_{pkql} \]

\[(R^\#)_{ijkl} = R_{ijpq} R_{pkql} - R_{ijpl} R_{pkql} \]

\[ l_{a,b}(R) = (1 + (m + 1)a) R_l + (1 + (m + 2)b) R_{Rico} + R_W \]

\[ D_{a,b} = l_{a,b}^{-1}((l_{a,b}(R))^2 + (l_{a,b}(R))^\#) - R^2 - R^\# \]

The main result from Ni-Wallach says:

\[ D_{a,b} = \left( \frac{m + 2}{2} b^2 + b - \frac{a}{2} \right) R_{c,d}^0 \bar{\omega} R_{c,d}^0 + \frac{a}{2} R_{c,d} \bar{\omega} R_{c,d} - b^2 ((R_{c,d}^0)^2) \bar{\omega} \]

\[ + \frac{|R_{c,d}^0|^2}{m(1 + (m + 1)a)} (mb^2(1 + 2b) - (a - 2b)(1 + 2b - mb^2)) I \]

Now we proceed to the proof of the lemma Chen-Sun-Tian using the above formula. Denote \( C \) to be the cone of curvature operators with nonnegative holomorphic bisectional curvature and \( R(t_0) \) the curvature evolved by the normalized Kähler-Ricci flow at time \( t = t_0 \), assume \( \bar{R} = R - \epsilon \bar{Ric} \bar{\omega} \) for some \( \epsilon > 0 \). Since by the assumption that \( Ric(R(t)) \) curvature has a uniform lower bound, it suffices to show that one can find a uniform constant \( c \) to make \( \bar{R} \) preserve its nonnegativity along the flow.

Note that one can find \( a \) and \( b \) such that \( \bar{R} = l_{a,b}(R) \), now we know \( R(0) \in Int C \) and we want \( l_{a,b}(R(t)) \in C \) for any time \( t \), which is equivalent to say \( R(t) \in l_{a,b}^{-1}(C) \). According to Hamilton’s maximum principle, it is enough show the new curvature cone \( l_{a,b}^{-1}(C) \) is preserved by

\[ \frac{d}{dt} R = R^2 + R^\# + R \]

which is the ODE counterpart of curvature evolution equation for the normalized Kähler-Ricci flow. If we further assume there exists \( c \) and \( d \) such that \( l_{c,d} = l_{a,b}^{-1} \), we only need to show

\[ l_{c,d}^{-1}((l_{c,d}(R))^2 + (l_{c,d}(R))^\# + l_{c,d}(R)) - R^2 - R^\# - R = D_{c,d}(R) \]

lies in the tangent cone of the invariant curvature cone \( C \) of nonnegative bisectional curvature. That is to say: \( D_{c,d}(R) \geq 0 \).

In next section we will carry out the detailed calculation to verify this.

2. Computations

Compare \( \bar{R} = R - \epsilon \bar{Ric} \bar{\omega} \) and \( l_{a,b}(R) = (1 + (m + 1)a) R_l + (1 + (m + 2)b) R_{Rico} + R_W \) to get \( a = -2\epsilon \), \( b = -\epsilon \).

From \( l_{a,b}(R) = (1 + (m + 1)a) R_l + (1 + (m + 2)b) R_{Rico} + R_W \) and \( l_{c,d} = l_{a,b}^{-1} \) one get:

\[ (1 + (m + 1)a)(1 + (m + 1)c) = 1 \]

\[ (1 + (m + 1)b)(1 + (m + 1)d) = 1 \]

Hence \( c = \frac{2\epsilon}{1 - 2(m + 1)\epsilon} \), \( d = \frac{-\epsilon}{1 - (m + 2)\epsilon} \), note that \( \xi > d > 0 \).

Now it is straightforward to get:
\[D_{c,d} = \left(\frac{m + 2}{2}d^2 + d - \frac{c}{2}\right)Rc^0 \nabla Rc^0 + \frac{c}{2}Rc^0 \nabla Rc - d^2((Rc^0)^2)\nabla \omega + \frac{|Rc^0|^2}{m(1 + (m + 1)c)}(md^2(1 + 2d) - (c - 2d)(1 + 2d - md^2))I\]

\[= \left(\frac{m + 2}{2}d^2 + d - \frac{c}{2}\right)(Rc^0 \nabla Rc - \frac{2s}{m}Rc^0 \omega + \frac{s^2}{m^2} \omega \nabla \omega) + \frac{c}{2}Rc^0 \nabla Rc - d^2((Rc^0)^2)\nabla \omega \]

\[+ \frac{|Rc^0|^2}{m(1 + (m + 1)c)}(md^2(1 + 2d) - (c - 2d)(1 + 2d - md^2))I\]

Since

\[\frac{c}{2} - d = \frac{\epsilon}{1 - 2(m + 1)c} - \frac{\epsilon}{1 - (m + 2)c} = \frac{m}{(1 - 2(m + 1)c)(1 - (m + 2)c)}\epsilon^2\]

Combining Perelman’s result on scalar curvature upper bound and Mok’s result that the holomorphic bisectional curvature preserving its nonnegativity we know uniform curvature upper bounds along the flow. Use the assumption that Ricci curvature has lower bound along the flow, it is easy to observe that the dominant terms in the above is \(\frac{|Rc^0|^2}{m}(1 + (m + 1)a)(md^2(1 + 2d) - (c - 2d)(1 + 2d - md^2))I\), which means that \(D_{c,d} \geq 0\).

References


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