An overview on PDE.

Example 1: Check $u(x, y) = x^2 - y^2$

Solves $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Example 2: Check $u(r, \theta) = r^2 \cos 2\theta$

Solves $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$.

Type of PDE:
- Linear PDE
- Nonlinear PDE

The general form of a linear second order PDE is given by:

$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$

A, B, C, D, E, F, G are either constants or functions of x and y.

When $G = 0$, we call it homogeneous.

PLAN:

1. Solutions to a linear PDE forms a vector space.
   - Superposition principle.

2. Classification of linear 2nd order constant coefficients PDE.
3. Three major examples and boundary value problems.

4. Some examples of solutions.

5. An example of nonlinear PDE. Navier-Stokes equations

1. Superposition principle states:
   If \( u_1, u_2, \ldots, u_k \) are solutions to a homogeneous linear PDE, then the linear combination
   \[ u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k \]
   where \( c_i \) are constants, is also a solution.

2. Classification of 2nd order linear PDE with constant coefficients:
   \[ A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \]
   Consider
   \[ \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix} \]
   \[ \det < 0 \iff 4AC - B^2 < 0 \] hyperbolic
   \[ \det = 0 \iff 4AC = B^2 \] parabolic
   \[ \det > 0 \iff 4AC \neq B^2 \] elliptic
Example: $3u_{xx} = u_{yy}$, parabolic
$u_{xx} = u_{yy}$, hyperbolic

3. Three major types: examples

- Vibration of the elastic string

$u_{tt} - c^2 u_{xx} = 0$
$0 \leq x \leq L$
$t > 0$

- Heat transfer of a rod

$u_t - k u_{xx} = 0$
$0 \leq x \leq L$
$t > 0$

- Equilibrium state of 2-dim heat transfer

$u_{xx} + u_{yy} = 0$
$0 \leq x \leq a$
$0 \leq y \leq b$

$\int u_{xx} + u_{yy} = 0$
$0 \leq x^2 + y^2 \leq R^2$
in polar coordinates
Since the general solution is not unique, we have to set up boundary & initial conditions (BVP)

For example, a standard BVP for wave equation

\[
\begin{align*}
U_{tt} - C^2 U_{xx} &= 0 & 0 < x < L \\ t > 0 \\
U(0, t) &= 0 \\
U(L, t) &= 0 \\
U(x, 0) &= f(x) & 0 \leq x \leq L \\
\frac{\partial U}{\partial t}(x, 0) &= g(x) & 0 \leq x \leq L
\end{align*}
\]

4. Travelling wave solution for wave equation

Def (Trapezoid)

\[
\begin{align*}
x &= a + ct \\
\Rightarrow t &= \frac{x}{c} - \frac{a}{c} \\
x &= b - ct \\
\Rightarrow t &= \frac{b}{c} - \frac{x}{c}
\end{align*}
\]
Thm 9.1

Every solution \( u(x,t) \) of

\[
U_{tt} - c^2 U_{xx} = 0 \quad \text{in the trapezoid}
\]

is of the form

\[
u(x,t) = f(x-ct) + g(x+ct)
\]

where \( f, g \) are two \( C^2 \) functions.

5 Nonlinear PDEs

Euler equations

Navier–Stokes equations

Wiki "Millennium Problems" by Clay Mathematics Institute.