Before we recall the exact statement of the Inverse Function Theorem, let’s think about what we’d like for it to say. We’ve been talking about solving equations. Naively, given a function \( f : \mathbb{R}^n \to \mathbb{R}^n \) and a value \( b \) in the range, we simply want to solve \( f(x) = b \). Newton’s method gives us a way to do this. But in the linear case, we have a much stronger situation: when \( f \) is invertible, we just have to find \( f^{-1} \) to get solutions to all equations \( f(x) = b \). By examples, we know that it’s generally hopeless to expect this to happen for non-linear functions. But if we know a solution exists for some \( b_0 \), we might hope that solutions also exist near \( b_0 \). The Inverse Function Theorem tells us that this hope is (often) justified, and that the solutions depend differentiably on \( b \) near \( b_0 \).

**Theorem 1** (Inverse Function Theorem). Let \( f : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) function on a neighborhood of \( x_0 \). Suppose that \( Df(x_0) \) is invertible. Then \( f \) has a local \( C^1 \) inverse on a neighborhood of \( x_0 \), i.e., \( f(x) = b \) has a solution for \( b \) in some ball around \( f(x_0) \).

The concept of “locally invertible” may be difficult. First, you should realize that a property being “local” on a set simply means that every point in that set is contained in a neighborhood on which the property holds. (As opposed to “pointwise”, which only has to hold at each point: continuity is an example of a pointwise property.) Some examples may help explain why local invertibility is such an important concept.

**Example 1** (A function that is everywhere locally invertible, but does not have a global inverse). Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^2 - 1 & \text{if } x < 0. \end{cases}
\]

At every point of \( \mathbb{R} \), \( f \) has a local inverse. For \( x > 0 \), it is \( y \mapsto \sqrt{y} \); for \( x < 0 \), it is \( y \mapsto -\sqrt{y} + 1 \). There is also an inverse on the interval \((-1, 1)\), given by

\[
y \mapsto \begin{cases} -\sqrt{y} + 1 & \text{if } y \in (-1, 0) \\ \sqrt{y} & \text{if } y \in [0, 1). \end{cases}
\]

However, \( f \) has no global inverse, because it is not one-to-one.

**Example 2** (A differentiable example). Consider the exponential function \( \exp : \mathbb{C} \to \mathbb{C} \). As you saw in an earlier homework, the derivative of \( \exp \) as a function \( \mathbb{R}^2 \to \mathbb{R}^2 \) at \( z_0 = x_0 + iy_0 \) is

\[
D \exp \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = e^{x_0} \begin{bmatrix} \cos y_0 & -\sin y_0 \\ \sin y_0 & \cos y_0 \end{bmatrix}.
\]

This matrix is always invertible—its determinant is \( e^{2x_0} \neq 0 \). Thus the Inverse Function Theorem guarantees a local inverse of \( \exp \) at each point of \( \mathbb{C} \), and the inverse will even be differentiable! (Aside: such a local inverse for \( \exp \) is called, naturally, a logarithm. But as we’ll see in a moment, logarithms are far from unique.)

However, \( \exp \) is not one-to-one on \( \mathbb{C} \): if \( z_1 = x + iy_1 \) and \( z_2 = x + iy_2 \), where \( y_1 \) and \( y_2 \) differ by a multiple of \( 2\pi \), then \( e^{z_1} = e^{z_2} \); \( \exp \) is periodic in the imaginary direction. (Wow!) Any point of \( \mathbb{C} \) is contained in a ball of radius \( \pi \) on which \( \exp \) is invertible. (In your spare time, you might think about what the image of this ball would look like.)