**MATH 3560 Groups and Geometry**
**Prelim 2 SOLUTIONS**

**Problem 1. (30 points)** This problem continues on the next page.

(a) Complete the following definitions.

(i) If $G$ is a group acting on the set $X$, and $g \in G$, then the *fixed-point set* of $g$ is ...

$$\mathrm{Fix}(g) = \{x \in X \mid g(x) = x\}$$

(ii) If $G$ is a group acting on the set $X$, and $x \in X$, then the *stabilizer* of $x$ is ...

$$\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}$$

(iii) A $2 \times 2$ matrix $A$ is *orthogonal* if ...

$A^\top A = I$, or equivalently if $A$ has the form $A = \begin{pmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$ (either both top signs or both bottom signs are to be taken together).

(iv) If $G$ is a group, $a \in G$ and $H$ is a subgroup of $G$, then the *left coset* $aH$ of $H$ is ...

$$aH = \{ah \in G \mid h \in H\}$$

(v) Two elements $f_1$ and $f_2$ of $\mathrm{Isom}(\mathbb{C})$ are *conjugate* if ...

there exists some $g \in \mathrm{Isom}(\mathbb{C})$ such that $f_2 = g \circ f_1 \circ g^{-1}$.

(vi) If $G$ and $H$ are groups and $\phi : G \to H$ is a homomorphism, then the *kernel* of $\phi$ is ...

$$\ker(\phi) = \{g \in G \mid \phi(g) = e_H\}$$

(b) List the four types of non-identity isometries of $\mathbb{C}$.

translations, rotations, reflections, and glide-reflections

(c) Which of the four types of isometries are *direct?*

translations and rotations

(d) Which of the four types of isometries have *no fixed points?*

translations and glide-reflections

(e) State Lagrange's Theorem.

If $G$ is a finite group, and $H$ is a subgroup of $G$, then the order of $H$ divides the order of $G$.

**Problem 2. (20 points)** For each of the statements below, write a T in the blank if the statement is true; write an F in the blank if the statement is false. If a statement is not true in all situations, you should count it as false. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

__F__ Any two rotations about the origin are conjugate.

__T__ Any two rotations by a clockwise angle of $\pi/3$ radians are conjugate.

__T__ Any two reflections in lines through the origin are conjugate.

__T__ The composition of two direct isometries is direct.

__F__ The composition of two opposite isometries is opposite.

__F__ If $a$, $b$, and $c$ are three points of $\mathbb{C}$, no two of which coincide, then there exists $f \in \mathrm{Isom}(\mathbb{C})$ such that $f(0) = a$, $f(1) = b$, and $f(i) = c$.

__F__ If $G$ is an infinite group then every subgroup of $G$ is infinite.

__F__ If $G$ is a finite subgroup of $\mathrm{Isom}(\mathbb{C})$ then $G$ contains at least one reflection.

__F__ If $\phi : G \to H$ is a homomorphism and $H$ is abelian, then $G$ is abelian.

__F__ If $\phi : G \to H$ is a homomorphism and $G$ is abelian, then $H$ is abelian.
(When I wrote this problem, I meant for $\phi$ to be onto, but neglected to include this condition. In that case, the statement would be true. Everyone will receive credit for it.)

**Problem 3. (30 points)** This problem consists of several unrelated parts, and continues on the next page.

(a) Let $b_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ and $b_2 = i$. Find an explicit element $g \in \text{Isom}(\mathbb{C})$ that conjugates $T_{b_1}$ to $T_{b_2}$. Justify your answer.

The rotation $R_{2\pi/3}$ by $2\pi/3$ radians about the origin works. We can check directly by computing (we use the facts that $b_1 = e^{-i\pi/6}$ and $b_2 = e^{i\pi/2}$): for all $z \in C$,

$$\begin{aligned}(R_{2\pi/3} \circ T_{b_1} \circ R_{-2\pi/3})(z) &= e^{i2\pi/3}(e^{-i2\pi/3}z + e^{-i\pi/6}) \\ &= z + e^{i\pi\left(\frac{2}{3}-\frac{1}{6}\right)} \\ &= z + e^{i\pi/2} \\ &= T_{b_2}(z)\end{aligned}$$

(b) Give at least two reasons that the rotation $R$ by $\pi/2$ radians around the origin and the reflection $M$ in the $x$-axis cannot be conjugate in $\text{Isom}(\mathbb{C})$. (Do not just state the definition of conjugate elements; use properties of an isometry that we have shown to be invariant under conjugation.)

Three reasons should be immediately apparent:

- $R$ is a direct isometry, while $M$ is opposite;
- $R$ has a single fixed point, while $M$ fixes an entire line;
- the order of $R$ is 4, while that of $M$ is 2.

All of these properties—direct/opposite, number of fixed points, order as a group element of $\text{Isom}(\mathbb{C})$—have been shown to be invariant under conjugation in $\text{Isom}(\mathbb{C})$. Any two of these suffice to answer the question. Other correct solutions may be possible.

(c) Let $g \in S_5$ be the permutation $(1345)(235)$.

(i) What are the orbits of $\langle g \rangle$, acting on the set $\{1, 2, 3, 4, 5\}$?

We find the orbits of $\langle g \rangle$ by calculating:

$$g(1) = 3, \qquad g(3) = 1,$$
$$g(2) = 4, \qquad g(4) = 5, \qquad g(5) = 2.$$

Therefore the orbits are $\{1, 3\}$ and $\{2, 4, 5\}$.

(ii) What is the order of $g$?

The computations in part (i) show that $g$ can be written $g = (13)(245)$. The order of $g$ equals the product of the orders of these (disjoint) cycles, so it is $2 \times 3 = 6$.

(d) List all subgroups of $(\mathbb{Z}_{10}, +)$. Prove that your answer is correct, naming any theorems you use.

By Lagrange's Theorem, we know that the order of any subgroup of $\mathbb{Z}_{10}$ must be 1, 2, 5, or 10. The first and last possibilities are trivial. The middle two, 2 and 5, are prime, and therefore any subgroup having one of these orders must be cyclic, and every non-identity element must be a generator. Therefore the subgroups are

$$\begin{aligned} \{0\} &= \langle 0 \rangle \\ \mathbb{Z}_{10} &= \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle \\ \{0, 2, 4, 6, 8\} &= \langle 2 \rangle = \langle 4 \rangle = \langle 6 \rangle = \langle 8 \rangle \\ \{0, 5\} &= \langle 5 \rangle \end{aligned}$$

We could also recall that any subgroup of a cyclic group must be cyclic, and enumerate the possibilities as above.

**Problem 4. (20 points)**

Recall that a subset $X \subseteq \mathbb{C}$ is *invariant* under $f \in \text{Isom}(\mathbb{C})$ if $f(z) \in X$ for all $z \in X$.

(a) Let $O_2$ be the "point group" of the origin. Give a description in words and write down matrices for all elements of $O_2$ under which the $x$-axis is invariant. What is the order of the group $G_0$ which these elements form? Is $G_0$ abelian?

An element of $O_2$ under which the $x$-axis is invariant must send $(1, 0)$ either to $(1, 0)$ or to $(-1, 0)$. There are therefore four possibilities for the matrices of these elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are, respectively, the identity, reflection in the $x$-axis, reflection in the $y$-axis, and rotation by $\pi$ about the origin. They form a group of order 4, and it is abelian, specifically it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Describe in words *all* isometries of the plane under which the $x$-axis is invariant. Let $G$ be the group they form. Is $G$ abelian? Justify your answer.

The isometries under which the $x$-axis is invariant consist of translations in the horizontal direction, glide-reflections of the form $T_b M_x$, where $M_x$ is reflection in the $x$-axis and $b$ is a horizontal vector (i.e., a real number), and conjugates of the elements given in part (a) by any of these translations or glide-reflections (i.e., we also get rotation by $\pi$ around any point of the $x$-axis, and reflection in any vertical line).

$G$ is not abelian: it contains, for example, rotation by $\pi$ about the origin and rotation by $\pi$ about the point $1 = (1, 0)$. Composing these in one direction gives the translation $T_2$, and in the other gives $T_{-2}$. Lots of other examples of non-commuting elements may be given.

(c) Draw or describe the orbit of $i \in \mathbb{C}$:

(i) under $G_0$;

The orbit is $\{i, -i\}$.

(ii) under all of $G$.

The orbit is the pair of lines having equations $y = 1$ and $y = -1$.