MATH 3560 Groups and Geometry
Final Examination Solutions

Problem 1. (20 points) Define the following.

(a) An *isometry* is a function $f : \mathbb{C} \to \mathbb{C}$ such that ...

$|f(z) - f(w)| = |z - w|$ for all $z, w \in \mathbb{C}$.

(b) Let X and Y be sets. A function $f : X \to Y$ is *one-to-one* if ...

Given any $z, w \in \mathbb{C}$, $f(z) = f(w)$ implies $z = w$;
OR, alternatively, whenever $z \neq w$, then $f(z) \neq f(w)$.

(c) Let X and Y be sets. A function $f : X \to Y$ is *onto* if ...

for every $y \in Y$ there exists some $x \in X$ such that $f(x) = y$.

(d) Let G and H be groups. A function $\phi : G \to H$ is a *homomorphism* if ...

$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for every $g_1, g_2 \in G$

(e) If G is a group and H is a subgroup of G, then H is a *normal* subgroup if ...

$gH = Hg$ for every $g \in G$; OR $gHg^{-1} = H$ for every $g \in G$;
OR $ghg^{-1} \in H$ for every $g \in G$, $h \in H$;
OR H is the kernel of a homomorphism with domain G; OR etc.

(f) A subset X of the plane $\mathbb{C}$ is *discrete* if ...

for every $x \in X$, there exists a distance δ such that for every $y \in X$, $y \neq x$ implies $|y - x| \geq \delta$.

(g) A subgroup G of $\mathrm{Isom}(\mathbb{C})$ is *discrete* if ...

the orbit of any point of $\mathbb{C}$ under G is a discrete subset of $\mathbb{C}$.

Problem 2. (20 points) For each of the statements below, write a T in the blank if the statement is true; write an F in the blank if the statement is false. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

_ T _ The set of all translations (including translation by the vector 0) forms a subgroup of Isom($\mathbb{C}$).

_ F _ The set of all rotations (including rotation by 0 radians) forms a subgroup of Isom($\mathbb{C}$).

_ F _ Let M_1 and M_2 be reflections in distinct lines L_1 and L_2, respectively. Then M_1 and M_2 do not commute.

_ T _ Let M_1 and M_2 be reflections in distinct lines L_1 and L_2, respectively. Then M_1 and M_2 are conjugate in Isom($\mathbb{C}$).

_ F _ If G is a subgroup of Isom($\mathbb{C}$) and G^+ is the subgroup of G consisting of direct isometries, then $[G : G^+] = 2$.

_ T _ If $n = p^k$, where p is prime and k is a positive integer, and G is a group of order n, then every element of G has order a power of p.

_ T _ If G is a group of order 15, and $\phi : G \to H$ is an onto homomorphism, then the order of H divides 15.

_ T _ If G is a subgroup of Isom($\mathbb{C}$) and $z \in \mathbb{C}$, then $\mathrm{Stab}_G(z)$ is normal in G if and only if G fixes z.

_ F _ If G is a discrete subgroup of Isom($\mathbb{C}$), then G contains a non-trivial translation.

_ T _ If G is a wallpaper group and $z \in \mathbb{C}$, then $\mathrm{Stab}_G(z)$ cannot have order 16.

Problem 3. (20 points) This problem consists of several unrelated parts.

(a) Use the definition of isometry to prove that every isometry $f : \mathbb{C} \to \mathbb{C}$ is one-to-one.

> Let $f : \mathbb{C} \to \mathbb{C}$ be a function that is not one-to-one, i.e., suppose there exist z and w in $\mathbb{C}$ such that $f(z) = f(w)$ but $z \neq w$. Then $|f(z) - f(w)| = 0$, but $|z - w| \neq 0$, and therefore f cannot be an isometry, because it does not preserve distances. This proves the contrapositive to the statement "every isometry of $\mathbb{C}$ is one-to-one", and hence this statement itself.

(b) Let G be a subgroup of Isom($\mathbb{C}$) of order 8.
(i) Name or describe in words two finite groups which fix the origin and to which G could be conjugate. Are either of these groups abelian?

> Two such groups are the group of rotations generated by a rotation through angle $\pi/4$ and the dihedral group of the square D_4. The first of these is abelian.

(ii) For which of the groups named in part (i) is there an orbit consisting of exactly 4 points? An orbit which has exactly 8 points? Are there any orbits with exactly 2 points? You may draw pictures to help justify your answers.

> Only the dihedral group has an orbit consisting of exactly 4 points; any point on one of the lines of reflection in D_4 that is not the origin will have an orbit with 4 points.
>
> Both groups have orbits in $\mathbb{C}$ that have exactly 8 points; for the group of rotations, any point other than the origin will have an orbit of size 8, while for the dihedral group, any point not on one of the lines of reflection or the bisector of two of these lines will have 8 points in its orbit.
>
> We have almost shown with the above that neither group has an orbit with exactly 2 points: in both cases, the orbit of the origin only contains one point, and most points will have an orbit with 8 points. If a point sits on one of the lines of reflection in D_4, then it is fixed by the corresponding reflection, and sent to a point of each of the (three) other lines of reflection, hence its orbit has 4 points. If a point sits on the bisector of a pair of reflections, then it has the same image by a reflection and a rotation, so it only has four points in its orbit.

Problem 4. (30 points) Let G be a group.
(a) Show that for all $g \in G$, the conjugation function $\phi_g : G \to G$, given by

$$\phi_g(h) = ghg^{-1}$$

for every $h \in G$, is an automorphism of G.

> ϕ_g is one-to-one: if $\phi_g(h_1) = \phi_g(h_2)$, this means that $gh_1g^{-1} = gh_2g^{-1}$, and by multiplying on the right by g and on the left by g^{-1}, we obtain $h_1 = h_2$.
>
> ϕ_g is onto: given $h \in G$, $g^{-1}hg$ is in G, and $\phi_g(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h$.
>
> Therefore ϕ_g is a bijection.
>
> ϕ_g is a homomorphism: $\phi_g(h_1h_2) = gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = \phi_g(h_1)\phi_g(h_2)$.
>
> Therefore ϕ_g is an automorphism of G.

(b) Let H be a subgroup of G, and suppose that $\phi : G \to G$ is an automorphism (not necessarily conjugation). Assume also that $\phi(H) = H$, where by definition

$$\phi(H) = \{\phi(h) \mid h \in H\}.$$

The *restriction* $\phi|_H : H \to H$ of ϕ to the subgroup H is then defined by

$$\phi|_H(h) = \phi(h)$$

for all $h \in H$. Verify that if $\phi : G \to G$ is an automorphism of G such that $\phi(H) = H$, then the restriction $\phi|_H : H \to H$ is an automorphism of H.

> $\phi|_H$ is a bijection: it is assumed that $\phi|_H$ is onto, and it must also be one-to-one because ϕ is.
>
> Because $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ for any $g_1, g_2 \in G$, we have
>
> $$\phi|_H(h_1h_2) = \phi(h_1h_2) = \phi(h_1)\phi(h_2) = \phi|_H(h_1)\,\phi|_H(h_2)$$
>
> for any $h_1, h_2 \in H$, and so $\phi|_H$ is also a homomorphism.
>
> Therefore $\phi|_H$ is an automorphism of H.

(c) A subgroup H of G is said to be a *characteristic* subgroup if for all automorphisms $\phi : G \to G$,

$$\phi(H) = H.$$

(i) Suppose that H is a characteristic subgroup of G. Use part (a) to prove that H is normal in G.

Let $g \in G$. We want to show that $gHg^{-1} = H$. Define ϕ_g as before; then ϕ_g is an automorphism of G. Hence $\phi_g(H) = H$, but this exactly translates to $gHg^{-1} = H$.

(ii) Suppose that H is a characteristic subgroup of G, and K is a characteristic subgroup of H. Use part (b) to prove that K is a characteristic subgroup of G.

Let ϕ be any automorphism of G. Because H is a characteristic subgroup of G, $\phi(H) = H$, and therefore by part (b), $\phi|_H$ is an automorphism of H. Because K is a characteristic subgroup of H, $\phi|_H(K) = K$. But for every $k \in K$, $\phi|_H(k) = \phi(k)$, and so $\phi(K) = K$, as well. Therefore K is a characteristic subgroup of G.

Problem 5. (30 points) This problem consists of several unrelated parts.

(a) Draw a finite figure P in the plane such that P is not invariant under any reflection (that is, for every reflection $M_L : \mathbb{C} \to \mathbb{C}$ we have $M_L(P) \neq P$), but there is an order 2 isometry $f : \mathbb{C} \to \mathbb{C}$ such that $f(P) = P$.

For example, the figure $\S$.

(b) Let $f \in \mathrm{Isom}(\mathbb{C})$. Suppose there are $b, b' \in \mathbb{C}$, $\theta, \theta' \in \mathbb{R}$ and $\epsilon, \epsilon' \in \{0, 1\}$ such that

$$f = T_b \circ R_\theta \circ M^\epsilon = T_{b'} \circ R_{\theta'} \circ M^{\epsilon'}.$$

(Here, T_b is translation by b, R_θ is rotation by θ radians counterclockwise about the origin, and M is the reflection in the x–axis; similarly for $T_{b'}$ and $R_{\theta'}$.)
(i) Prove that $\epsilon = \epsilon'$, by considering direct and opposite isometries.

T_b, $T_{b'}$, R_θ, and $R_{\theta'}$ are all direct isometries. We know that if either expression yields an opposite isometry, then both must. In this case, $\epsilon = \epsilon' = 1$. In order for both sides to be direct, we must have $\epsilon = \epsilon' = 0$.

(ii) Prove that $b = b'$ and $\theta = \theta'$.

We evaluate both expressions for f at 0:

$$(T_b \circ R_\theta \circ M^\epsilon)(0) = T_b(R_\theta(M^\epsilon(0))) = T_b(R_\theta(0)) = T_b(0) = b$$

$$(T_{b'} \circ R_{\theta'} \circ M^\epsilon)(0) = T_{b'}(R_{\theta'}(M^\epsilon(0))) = T_{b'}(R_{\theta'}(0)) = T_{b'}(0) = b'$$

and therefore we have $b = f(0) = b'$.

Now we have $T_b R_\theta M^\epsilon = T_b R_{\theta'} M^\epsilon$, and therefore, using the group structure of $\mathrm{Isom}(\mathbb{C})$, $R_\theta = R_{\theta'}$. This means $e^{i\theta} z = e^{i\theta'} z$ for all $z \in \mathbb{C}$, which implies $\theta = \theta'$.

(c) Let R_{z_1,θ_1} and R_{z_2,θ_2} be two non-trivial rotations with distinct centers, that is, $z_1 \neq z_2$. Show that the group generated by R_{z_1,θ_1} and R_{z_2,θ_2} contains a translation.
(**Hint:** show that the group contains a rotation of the form $R_{z_0,-\theta_1}$, where $z_0 \neq z_1$. Also, you do not need to find the precise translation vector.)

Let $z_0 = R_{z_2,\theta_2}(z_1)$. Then $z_0 \neq z_1$ because $z_2 \neq z_1$, and

$$R_{z_2,\theta_2} \circ R_{z_1,\theta_1}^{-1} \circ R_{z_2,\theta_2}^{-1} = R_{z_0,-\theta_1}.$$

The composition $R_{z_1,\theta_1} \circ R_{z_0,-\theta_1}$ is a non-trivial translation: it is a direct isometry, its derivative is the composition $R_{0,\theta_1} \circ R_{0,-\theta_1} = \mathrm{id}$, and it is not the identity, because z_0 is not fixed.

Problem 6. (30 points) This problem continues on the next page.

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *unitary* if for all $z, w \in \mathbb{C}$

$$\overline{f(z)}f(w) = \bar{z}w.$$

(a) Show that every rotation around the origin is unitary.

A rotation R around the origin is given by $z \mapsto e^{i\theta}z$ for some θ. Then we have

$$\overline{R(z)}R(w) = (\overline{e^{i\theta}z})e^{i\theta}w = e^{-i\theta}\bar{z}e^{i\theta}w = e^{0}\bar{z}w = \bar{z}w$$

by properties of conjugation and the commutativity of multiplication. Therefore R is unitary.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a unitary function. Prove that for every $z \in \mathbb{C}$,

$$|f(z)| = |z|.$$

By the definition of a unitary function, we have

$$\overline{f(z)}f(z) = \bar{z}z.$$

The left-hand side equals $|f(z)|^2$, and the right-hand side equals $|z|^2$. Thus we have $|f(z)|^2 = |z|^2$, and because both sides are positive, we can take square roots to get $|f(z)| = |z|$.

(c) Show that every unitary function $f : \mathbb{C} \to \mathbb{C}$ is an isometry.

Suppose f is unitary. Let $z, w \in \mathbb{C}$. Then

$$|f(z) - f(w)|^2 = \overline{(f(z) - f(w))}(f(z) - f(w)) = (\overline{f(z)} - \overline{f(w)})(f(z) - f(w))$$

by properties of conjugation. Expanding, we obtain

$$\overline{f(z)}f(z) - \overline{f(z)}f(w) - \overline{f(w)}f(z) + \overline{f(w)}f(w).$$

Now we apply the definition of unitary function to each term so that this sum becomes

$$\bar{z}z - \bar{z}w - \bar{w}z + \bar{w}w = (\bar{z} - \bar{w})(z - w) = \overline{(z - w)}(z - w) = |z - w|^2.$$

As in part (b), this implies that $|f(z) - f(w)| = |z - w|$.

Therefore f is an isometry.

(d) Show that every unitary function $f : \mathbb{C} \to \mathbb{C}$ is a rotation around the origin. You may use previous parts of this problem, even if you did not complete them.

By part (c), f is an isometry. By part (b), $|f(0)| = |0| = 0$, and therefore 0 is a fixed point of f. Hence, by the classification of isometries, f must be either a rotation about the origin or a reflection in a line through the origin.

We must eliminate the possibility that f is a reflection: there are at least two ways to do this.

- Suppose that f is a reflection in a line L, which must pass through the origin. Then f has the form $f(z) = e^{2i\theta}\bar{z}$ for some θ. This leads to $\overline{f(z)}f(z) = e^{-2i\theta}ze^{2i\theta}z = z^2$, which does not equal $|z|^2$ if z is not real. Therefore f cannot be a reflection.
- Consider the imaginary part of the product $\bar{z}w$: if we set $z = x + iy$ and $w = u + iv$, then

  $$\operatorname{Im}(\bar{z}w) = \operatorname{Im}(x - iy)(u + iv) = \operatorname{Im}\big((xu + yv) + i(xv - yu)\big) = xv - yu,$$

  which is the determinant of z and w, taken as vectors in $\mathbb{R}^2$. Because f is unitary, we must have in particular that $\operatorname{Im}(\overline{f(z)}f(w)) = \operatorname{Im}(\bar{z}w)$, i.e., f preserves the determinant, so f must be direct.

Either line of reasoning leads to the conclusion that f is a rotation.