1. [TP 3.15] Consider a voting system for the six New England states where there are a total of seventeen votes and twelve or more are required for passage. Votes are distributed as follows:
   MA:4 ME:3 NH:2 CT:4 RI:3 VT:1
   (a) Calculate $SSI(MA)$.
   
   Solution. $SSI(MA) = \frac{168}{720}$
   
   (b) Calculate $SSI(ME)$.
   
   Solution. $SSI(ME) = \frac{132}{720}$
   
   (c) Calculate $SSI(NH)$.
   
   Solution. $SSI(NH) = \frac{96}{720}$
   
   (d) Calculate $SSI(VT)$.
   
   Solution. $SSI(VT) = \frac{24}{720}$


   Solution. Here are the Banzhaf power and indices of the New England states:
   
   \[
   \begin{align*}
   TBP(MA) &= 10, BI(MA) = \frac{10}{44} \\
   TBP(CT) &= 10, BI(CT) = \frac{10}{44} \\
   TBP(ME) &= 8, BI(ME) = \frac{8}{44} \\
   TBP(RI) &= 8, BI(RI) = \frac{8}{44} \\
   TBP(NH) &= 6, BI(NH) = \frac{6}{44} \\
   TBP(VT) &= 2, BI(CT) = \frac{2}{44}
   \end{align*}
   \]

2. [TP 3.20] Suppose we have a yes-no voting system with four voters $A, B, C,$ and $D$, where the winning coalitions are as follows:
   
   \[
   ABCD, ABC, ABD, ACD, BCD, AB, AD
   \]

   (a) Compute the Banzhaf index of each voter.

   Proof. These are the Banzhaf indices:
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\[
BI(A) = \frac{5}{12} \\
BI(B) = \frac{3}{12} \\
BI(C) = \frac{1}{12} \\
BI(D) = \frac{3}{12}
\]

(b) Compute the Shapley-Shubik index of each voter.

**Proof.** These are the Shapley-Shubik indices:

\[
SSI(A) = \frac{10}{24} \\
SSI(B) = \frac{6}{24} \\
SSI(C) = \frac{2}{24} \\
SSI(D) = \frac{6}{24}
\]

\[\boxed{}\]

3. [TP 3.21] Suppose that the New England states (ME, MA, RI, CT, VT, NH) decide to vote amongst themselves on a particular issue. They use their electoral votes above to create a six-state weighted voting system. The quota is set at 18.

(a) Calculate the Banzhaf index of each state.

**Solution.** The Banzhaf indices are:

\[
BI(ME) = \frac{6}{56} \\
BI(MA) = \frac{22}{56} \\
BI(RI) = \frac{6}{56} \\
BI(CT) = \frac{10}{56} \\
BI(VT) = \frac{6}{56} \\
BI(NH) = \frac{6}{56}
\]

\[\boxed{}\]

(b) Calculate the Shaley-Shubik index of each state.

**Solution.** These are the Shapley-Shubik indices:

\[
SSI(ME) = \frac{72}{720} \\
SSI(MA) = \frac{288}{720}
\]
\[
\begin{align*}
SSI(RI) &= \frac{72}{720} \\
SSI(CT) &= \frac{144}{720} \\
SSI(VT) &= \frac{72}{720} \\
SSI(NH) &= \frac{72}{720}
\end{align*}
\]

4. [TP 3.24] Consider the mini-federal system with thirteen voters wherein there are six House members, six senators, and the president, and passage requires half the House, half the Senate, and the president, or two-thirds of both houses.

(a) Calculate the Banzhaf index of the president.

Solution. These are the powers and indices:

\[
\begin{align*}
TBP(\text{President}) &= \binom{9}{3} \times \binom{9}{3} + \binom{9}{4} \times \binom{9}{3} + \binom{9}{5} \times \binom{9}{3} + \binom{9}{6} \times \binom{9}{3} + \binom{9}{7} \times \binom{9}{3} + \binom{9}{8} \times \binom{9}{3} + \binom{9}{9} \times \binom{9}{3} \\
TBP(\text{Housemember}) &= \binom{8}{2} \times \binom{8}{5} + \binom{8}{3} \times \binom{8}{4} + \binom{8}{5} \times \binom{8}{4} + \binom{8}{6} \times \binom{8}{4} + \binom{8}{8} \times \binom{8}{4} + \binom{8}{9} \times \binom{8}{4} \\
TBP(\text{Senator}) &= \binom{5}{2} \times \binom{5}{6} + \binom{5}{3} \times \binom{5}{5} + \binom{5}{4} \times \binom{5}{5} + \binom{5}{5} \times \binom{5}{5} \\
BI(\text{President}) &= \frac{TBP(\text{President})}{TBP(\text{President}) + 6 \times TBP(\text{Housemember}) + 6 \times TBP(\text{Senator})}
\end{align*}
\]

(b) Calculate the Shapley-Shubik index of the president.

Solution. \( SSI(\text{President}) = \frac{\binom{6}{3} \times \binom{6}{3} \times 6!6! + \binom{6}{4} \times \binom{6}{3} \times 7!5! + \binom{6}{5} \times \binom{6}{3} \times 8!4! + \binom{6}{6} \times \binom{6}{3} \times 9!3! + \binom{6}{7} \times \binom{6}{3} \times 9!3!}{13!} \) 

5. [TP 3.33] (a) Discuss whether or not you find it paradoxical that in going from the system \([8 : 5, 3, 1, 1, 1]\) to the system \([8 : 4, 4, 0, 0, 0]\), the first player’s Banzhaf power increases.

Solution. Yes, it does seem paradoxical that player one loses some of his votes and yet is able to increase his Banzhaf power.

(b) Discuss whether or not you find it paradoxical that two different choices of weights, such as \([8 : 4, 7, 0, 0, 0]\) and \([8 : 4, 4, 1, 1, 1]\) give the same yes-no voting system.

Solution. Yes, it does seem paradoxical that player two is able to distribute his votes to three other players and yet there is no change in the yes-no voting system.

(c) What, if anything, do your responses to (a) and (b) say about the Felsenthal–Machover paradox?

Solution. This illustrates that Banzhaf power is vulnerable to the Felsenthal-Machover paradox. (i.e. changes in the system affect the voting powers in a way which seems counter-intuitive.)
6. [TP 3.34] The Shapley–Shubik index is not without paradoxical aspects. For example, the following was pointed out by William Zwicker. Suppose we have a bicameral yes–no voting system wherein an issue must win in both the House and the Senate in order to pass. (We are not assuming that the House and Senate necessarily use majority rule, but we are assuming they have no common members.) Suppose that both you and Fred belong to the House and that—when the House is considered as a yes–no voting system in its own right Fred has three times as much “power” as you have. Then shouldn’t Fred still have three times as much power as you have when we consider the bicameral yes-no voting system? (This is a rhetorical question.)

(a) Suppose that $X_1, \ldots, X_m$ are the winning coalitions in the House and $Y_1, \ldots, Y_n$ are the winning coalitions in the Senate. Assume that Fred belongs to $t$ of the winning coalitions in the House and that you belong to $z$ of the winning coalitions in the House.

1. Show that there are $mn$ winning coalitions in the bicameral system.

**Solution.** The winning coalitions in the bicameral system are all tuples of the form $\{X_i, Y_j\}$ where $i$ ranges from 1 to $m$ and $j$ ranges from 1 to $n$. Hence, by the multiplication principle, there are $m \times n = mn$ winning coalitions in the bicameral system.

2. Use Procedure 2 to show that Fred’s total Banzhaf power in the House is $2t - m$ and that yours is $2z - m$.

**Solution.**

Fred’s total Banzhaf power in the House is:

$$\text{TBP(Fred)} = 2 \times \text{(Number of winning coalitions Fred is a part of)} - \text{total number of winning coalitions} = 2t - m$$

Your total Banzhaf power in the House is:

$$\text{TBP(You)} = 2 \times \text{(Number of winning coalitions you are a part of)} - \text{total number of winning coalitions} = 2z - m$$

3. Use Procedure 2 to show that Fred’s total Banzhaf power in the bicameral system is $2tn - mn$ and that yours is $2zn - mn$.

**Solution.**

Fred’s total Banzhaf power in the bicameral system is:

$$\text{TBP(Fred)} = 2 \times \text{(Number of winning coalitions Fred is a part of)} - \text{total number of winning coalitions} = 2tn - mn$$

This is because there are a total of $mn$ winning coalitions in the bicameral system and Fred is part of those tuples $\{X_i, Y_j\}$ where Fred belongs to $X_i$. Hence there are $t \times n$ of those.

Your total Banzhaf power in the bicameral system is:

$$\text{TBP(You)} = 2 \times \text{(Number of winning coalitions you are a part of)} - \text{total number of winning coalitions} = 2zn - mn$$

This is because there are a total of $mn$ winning coalitions in the bicameral system and you are part of those tuples $\{X_i, Y_j\}$ where you belong to $X_i$. Hence there are $z \times n$ of those.

4. Show that if Fred has $v$ times as much power (as measured by the Banzhaf index) in the House as you have, then Fred also has $v$ times as much power (as measured by the Banzhaf index) in the bicameral system as you have.

**Solution.** Fred has $v$ times as much power in the house as you

$$\implies v = \frac{2t - m}{2z - m}$$

The ratio of Fred’s power to your power in the bicameral system is

$$\frac{2tn - mn}{2zn - mn} = \frac{2t - m}{2z - m} = v.$$
(b) Suppose the House consists of you, Fred, and Bill, and suppose there are two minimal winning coalitions in the House: Fred alone (as one), and you and Bill together (as the other). In the Senate, there are two people and each alone is a minimal winning coalition.

1. Show that in the House alone, Fred has four times as much power—as measured by the Shapley-Shubik index—as you have.

Solution. \(SSI(Fred) = \frac{4}{3!}\)
\[SSI(you) = \frac{1}{3!}\]
Hence Fred has four times as much power in the house. \(\square\)

2. Show that in the bicameral system, this is no longer true.

Solution. \(SSI(Fred) = \frac{44}{5!}\)
\[SSI(you) = \frac{14}{5!}\]
In the bicameral system Fred has less than four times your power. \(\square\)