1. [TP 1.1] Let $X$ and $Y$ both be the set of non-negative integers $0, 1, 2, 3, \ldots$. Determine if the given procedures are functions from $X$ to $Y$.

(a) The procedure corresponding to taking the square root of the input.

Solution. No, since the square root of a non-negative integer is not necessarily a non-negative integer. For example, the square root of 3 does not lie in $Y$. □

(b) The procedure corresponding to doubling the input.

Solution. Yes, because given any $x \in X$, the product $2x$ is a non-negative integer and so $2x \in Y$. □

(c) The procedure that, given input $x$, outputs $y$ if and only if $y$ is two units away from $x$ on the number line.

Solution. No, because it does not assign a unique object of $Y$ to every given element of $X$. For example, consider the element $3 \in X$, the procedure outputs both 1 and 5; while these are individually elements of $Y$, the output “1 and 5” is not an element of $Y$, so this procedure is not a function. □

(d) The procedure that, given input $x$, outputs the number 17.

Solution. Yes, because this assigns to each object in $X$ a unique object in $Y$. □

For problems 2–4, you don’t need to show all the calculations (but feel free to attach them), but make a neat table that summarizes the results for each voting method.

2. [TP 1.4]

Solution. Condorcet: No Condorcet winner.

Plurality: No winner (5-way tie)

Borda: c wins

Hare: All have the same number of first place votes, so it’s a 5-way tie.

Sequential Pairwise Voting with a Fixed Agenda: b wins.

Dictatorship: e wins. □

3. [TP 1.5]

Solution. (a) Condorcet’s method: b wins.

(b) plurality: a wins.

(c) Borda count: b wins.

(d) Hare system: b wins.

(e) approval voting: Tie between a and d.

(f) If there is a Condorcet winner, that is the social choice; otherwise, plurality: b wins, as the Condorcet winner. □
4. [TP 1.6]

Solutions. (a) Condorcet’s method: c wins, as the Condorcet winner.
(b) plurality: c wins.
(c) Borda count: c wins.
(d) Hare system: c wins.
(e) approval voting: c wins.
(f) If there is a Condorcet winner, that is the social choice; otherwise, Hare system: c wins, as the Condorcet winner.

5. [TP 1.8] Prove or disprove each of the following:
(a) Plurality voting always yields a unique social choice. [Note that “social choice” means that there is a single winner, that is, no ties. (cf. p.3)]
   Proof. False. Consider a population of two voters (with votes of equal weight) who vote for two different candidates as their respective first choices. Plurality then results in a tie, which is not a social choice, but rather, a social choice set.

(b) The Borda count always yields a unique social choice.
   Proof. False. Consider the same voting population as the last problem, and suppose that there are two candidates A and B running, and the two voters submit ranked ballots with (A, B) and (B, A) respectively. Then the Borda count for both A and B are 1, and so we have a tie once again, so not a social choice.

(c) The Hare system always yields a unique social choice.
   Proof. False. Consider again the same situation as in part (b). Each candidate has the same number of first place votes, and so we cannot eliminate either candidate. Thus, the Hare system does not result in a unique social choice in this situation either.

(d) Sequential pairwise voting with a fixed agenda always yields a unique social choice.
   Proof. False. Consider again the same situation as in part (b). There is no winner and there is no third alternative against whom the first two candidates can be pitted against. Thus, we do not obtain a unique social choice here, either.

(e) A dictatorship always yields a unique social choice.
   Proof. True. The dictator’s ranked list must have a unique candidate in the first place position, and this is the social choice under the dictatorship social choice procedure.

6. [TP 1.9]
(a) Find the social choice using the Borda count.
   Solution. a wins (with 7 points).

(b) Suppose we change the way we assign points so that first place is worth 8 points, second place is worth 4 points, third place is worth -4 points, and fourth place is worth -8 points. Redo the Borda procedure using these new numbers.
   Solution. a wins (with 8 points).

(c) Redo (b) using the points $-1, -5, -9, -13$ for (respectively) first, second, third, and fourth place.
   Solution. a wins (with -24 points).
(d) Do as in (c) using 9, 4, 1, and 0 points for (respectively) first, second, third, and fourth place.

Solution. \(c\) wins (with 18 points).

(e) Propose a condition on the way points are assigned that is sufficient the guarantee that the winner is the same as the Borda winner with points assigned the usual way.

Solution. If the points are scaled by a constant and added to a (possibly different) constant, then the winner is the same as the Borda winner with points assigned the usual way.