Part III - Category Theory

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1 Definitions and Examples

1.1 Definition. A category C consists of

(i) A collection $\mathrm{ob}\,\mathcal{C}$ of objects $A, B, C, \ldots$

(ii) A collection $\mathrm{mor}\,\mathcal{C}$ of morphisms $f, g, h, \ldots$

(iii) A rule assigning to each $f \in \mathrm{mor}\,\mathcal{C}$ two objectives $\mathrm{dom}\, f$ and $\mathrm{cod}\, f$, its domain and codomain. We write $f : A \to B$. or $A \xrightarrow{f} B$ for 'f is a morphism with $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$.'

(iv) For each pair (f, g) of morphisms with $\mathrm{dom}\, f = \mathrm{cod}\, g$, we have a composite morphism $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$ subject to the axioms that $1_B f = f = f 1_A$ for any $f : A \to B$ and $(hg)f = h(gf)$ where hg and gf are defined.

1.2 Remark.

1. The definition does not depend on set theory. In the context of set theory, we say that $\mathcal{C}$ is a small category if $\mathrm{ob}\,\mathcal{C}$ and $\mathrm{mor}\,\mathcal{C}$ are sets.

2. We could eliminate objects from the definition by identifying them with the identity morphisms.

1.3 Examples.

(a) The category of **Set** has all sets as objects and functions as morphisms (actually, morphisms are triples (B, f, A) where $f \in A \times B$ is a function in the set theoretic sense).

(b) The categories of **Gp** of groups, **Rng** of rings and $\mathbf{Mod}_R$ of R-modules have sets with algebraic structures as objects and homomorphisms as morphisms.

(c) The category of **Top** of topological spaces and continuous maps, **Met** of metric spaces and lipshitz maps and **Diff** of differential manifolds and smooth maps.

(d) The category **Htpy** has the same objects as **Top** but the morphisms are homotopy classes of functions. More generally we can factor out any equivalence relation on morphisms such that if $f \simeq g$ we have $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$, $fh \simeq gh$ and $kf \simeq kg$ wherever defined.

(e) Given any category $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$ but dom and cod are interchanged.

(f) A small category with one object is a monoid. In particular every group can be considered as a category with one object in which every morphism is an isomorphism.

(g) A groupoid us a category in which every morphism is an isomorphism. For example, the fundamental groupoid $\pi(X)$ of a space with points as objects and homotopy classes of paths as morphisms.

(h) A discrete category is one whose only morphisms are identities (so a small discrete category is a set). A preorder is a category with at most one morphism from A to B for any objects A, B. Equivalently it is a collection of objects with a reflexive, transitive relation, so a poset is a small preorder whose only isomorphisms are identities. An equivalence relation is a preorder which is also a groupoid.

(i) The category **Rel** has sets as objections and morphisms $A \to B$ are relations, arbitrary subsets $R \subset B \times A$. Composition $S \circ R \subset C \times B$:

$$S \circ R = \{(c, a) | (\exists b \in B)((c, b) \in S \wedge (b, a) \in R)\}.$$

Contains **Set** as a subcategory and also **Part** of sets and partial functions (a partial function $A \to B$ is a function on some subset of A).

(j) Let k be a field. The category $\mathbf{Mat}_k$ has the natural numbers as objects and morphisms $n \to m$ are $m \times n$ matrices with entries in k. Composition is matrix multiplication. Alternatively, given a lattice L, we can form $\mathbf{Mat}_L$, whose morphisms are matrices of elements of L, with multiplication given by $[(l_{ij})(m_{jk})]_{ik} = \vee_j (l_{ij} \wedge m_{jk})$.

(k) Given a theory in a formal system, the category $\mathbf{Der}_T$ has formula of the formal language as objects and morphisms $\varphi \to \psi$ are derivations of ψ from φ. Composition is concatenation.

1.4 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of

(i) a mapping $A \to FA : \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$

(ii) a mapping $f \to Ff : \operatorname{mor} \mathcal{C} \to \operatorname{mor} \mathcal{D}$ such that

- $\operatorname{dom} Ff = F \operatorname{dom} f$
- $\operatorname{cod} Ff = F \operatorname{cod} f$

- $F1_A = 1_{FA}$
- $F(gf) = (Fg)(Ff)$ where gf is defined.

1.5 Examples.

(a) We have a functor $U : \mathbf{Gp} \to \mathbf{Set}$ sending a group to its underlying set and a group homomorphism to its underlying function. Similarly for **Top**, etc.. We also have $U : \mathbf{Rng} \to \mathbf{Gp}$. These are called forgetful functors.

(b) There is a functor $F : \mathbf{Set} \to \mathbf{Gp}$ sending a set A to the free group FA generated by A and a function $A \xrightarrow{f} B$ to the unique homomorphism $Ff : FA \to FB$ sending each generator $a \in A$ to $f(a) \in B \subset FB$.

(c) We have a functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ sending A to its power set $\mathcal{P}A$ and $f : A \to B$ to the mapping $\mathcal{P}f : \mathcal{P}A \to \mathcal{P}B$ sending a subset $A' \subset A$ to $f(A')$. But we also have $\mathcal{P}^* : \mathbf{Set} \to \mathbf{Set}^{\mathbf{op}}$ defined by $\mathcal{P}^*A = \mathcal{P}A$ and $\mathcal{P}^*f(B') = f^{-1}(B')$. By a contravariant functor $\mathcal{C} \to \mathcal{D}$ we mean a functor $\mathcal{C} \to \mathcal{D}^{\mathbf{op}}$ (we can also speak of covariant).

(d) We have a functor $(-)^* : \mathbf{Mod}_k^{\mathbf{op}} \to \mathbf{Mod}_k$ sending a vector space V over k to its dual.

(e) We write **Cat** for the (large) category of small categories and the functions between them. Then $\mathcal{C} \to \mathcal{C}^{\mathbf{op}}$ defines a functor $\mathbf{Cat} \to \mathbf{Cat}$ with $F^{\mathbf{op}}$ defined to be F.

(f) A functor between monads is a homomorphism, etc...

(g) A functor between posets is an order preserving function.

(h) Let G be a group, considered as a category. A functor $F : G \to \mathbf{Set}$ is a set $A = F*$ equipped with an action of G, i.e. a permutation representation of G. Similarly for any field k, a functor $G \to \mathbf{Mod}_k$ is just a k-linear representation of G.

(i) We have functors $\pi_n : \mathbf{Htpy}_* \to \mathbf{Gp}$ ($\mathbf{Htpy}_*$ is the category of pointed topological spaces and basepoint respecting homotopy classes of basepoint preserving continuous maps). π_n sends (X, x) to its n-th homotopy group. Similarly, we have functors $H_n : \mathbf{Htpy} \to \mathbf{AbGp}$ sending X to its n-th homology group.

1.6 Definition. Let $\mathcal{C}$, $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ two functors. A natural transformation $\alpha : F \to G$ consists of a mapping $A \to \alpha_A$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$ such that $\alpha_A : FA \to GA$ for any A and

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

commutes for every $f : A \to B$ in $\mathcal{C}$. These obviously compose, so we have a category $[\mathcal{C}, \mathcal{D}]$ of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

1.7 Examples.

(a) Let k be a field. The double dual operation $V \to V^{**}$ defines a covariant functor $\mathbf{Mod}_k \to \mathbf{Mod}_k$. For every V we have a canonical mapping $\alpha_V : V \to V^{**}$ sending $x \in V$ to the mapping $\varphi \to \varphi(x)$. The α_V's are the components of a natural transformation $1_{\mathbf{Mod}_k} \to (-)^{**}$. If we restrict to the category of $\mathbf{fdMod}_k$, of finite dimensional vector spaces, then the α_V's are isomorphisms for all V. This implies that α is an isomorphism in $[\mathbf{fdMod}_k, \mathbf{fdMod}_k]$.

(b) Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the (covariant) power set functor of 1.5c. There is a natural transformation $\eta : 1_{\mathbf{Set}} \to \mathcal{P}$ such that $\eta_A : A \to \mathcal{P}A$ sends each $a \in A$ to $\{a\}$.

(c) Let G, H be groups, $f, g : G \to H$ two homomorphisms. What is a natural transformation $\alpha : f \to g$? It defines an element $y = \alpha_*$ of H such that for any $x \in G$, we have $yf(x) = f(x)y$, $g(x) = yf(x)y^{-1}$.

(d) For every pointed space (X, x) and $n \geq 1$, there is a canonical mapping $h_n : \pi_n(X, x) \to H_n(X)$ (the Hurewicz homomorphism). This is a natural transformation from $\pi_n : \mathbf{Htpy}_* \to \mathbf{Gp}$ to the composition $\mathbf{Htpy}_* \to \mathbf{Htpy} \xrightarrow{H_n} \mathbf{AbGp} \to \mathbf{Gp}$.

1.8 Definition. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(i) Say F is faithful if given any two objects A, B of $\mathcal{C}$ and two morphisms $f, g : A \to B$, $Ff = Fg$ implies $f = g$.

(ii) We say F is full if, given two objects A, B of $\mathcal{C}$ any morphism $FA \xrightarrow{g} FB$ in $\mathcal{D}$ is of the form Ff for some $f : A \to B$ in $\mathcal{C}$.

(iii) We say that a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is full if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor. Ex. $\mathbf{AbGp} \to \mathbf{Gp}$ is full.

1.9 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$.

1.10 Lemma. *[Assuming AC] A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence iff it is full, faithful and essential surjective on objects (i.e. every $B \in \operatorname{ob}\mathcal{D}$ is isomorphic to some FA).*

Proof. Suppose we have G, α, β as above. For any $B \in \operatorname{ob}\mathcal{D}$, we have $B \cong FGB$, so F is essentially surjective. Suppose that we have $A \underset{f}{\overset{g}{\rightrightarrows}} B$ in $\mathcal{C}$ with $Ff = Fg$, then $GFf = GFg$, so

$$f = \alpha_B^{-1}(GFf)\alpha_A = \alpha_B^{-1}(GFg)\alpha_A = g$$

so F is faithful.

Suppose $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$, $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. Suppose that $A, A' \in \operatorname{ob}\mathcal{C}$ and $g : FA \to FA'$ in $\mathcal{D}$. Let f be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFA' \xrightarrow{\alpha_{A'}^{-1}} A',$$

then $GFf = Gg$ since both morphisms make the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & A' \arrow[d, "\alpha_{a'}"] \\ GFA \arrow[r, "GFf"'] & GFA' \end{tikzcd}$$

commute. But G is faithful since it is part of an equivalence, so $Ff = g$.

Conversely, suppose that F is full, faithful and essentially surjective. For each $B \in \operatorname{ob}\mathcal{D}$ pick $A \in \operatorname{ob}\mathcal{C}$ such that $\beta_B : FA \to B$ is an isomorphism. Write GB for A. Given $g : B \to B'$, we have a composite

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

which must be Ff for a unique $f : GB \to GB'$. Define f to be Gg. Given $g' : B' \to B''$ the morphisms $(Gg')(Gg)$ and $G(gg')$ have the same image under F, thus they are equal. Thus G is a functor and β is a natural transformation from $FG \to 1_{\mathcal{D}}$. We know that $FGFA \xrightarrow{\beta_{FA}} FA$ is an isomorphism, so β_{FA}^{-1} is of the form $F\alpha_A$ for a unique $\alpha_A : A \to GFA$ which must be an isomorphism by the faithfulness of F. Given $f : A \to A'$ in $\mathcal{C}$, the composite $\alpha_{A'} f$ and $(GFf)\alpha_A$ have the same image under F by naturality of β^{-1} and they are equal. $\square$

1.11 Examples.

(a) Given a category $\mathcal{C}$ and a particular object B of $\mathcal{C}$ we write $\mathcal{C}/B$ for the category whose objects are morphisms $A \xrightarrow{f} B$ with codomain B in $\mathcal{C}$ and whose morphisms are commutative triangles

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

in $\mathcal{C}$. For $\mathcal{C} = \mathbf{Set}$ we have an equivalence of categories $\mathbf{Set}/B \cong \mathbf{Set}^B$. The functor $F : \mathbf{Set}/B \to \mathbf{Set}^B$ sends $A \xrightarrow{f} B$ to $(f^{-1}(b)|b \in B)$ and $G : \mathbf{Set}^B \to \mathbf{Set}/B$ sends $(A_b|b \in B)$ to $\bigcup(A_b \times \{b\})$ mapping to B by second projection.

(b) The coslice category $B/\mathcal{C}$ is defined as $(\mathcal{C}^{\mathbf{op}}/B)^{\mathbf{op}}$. In particular $1/\mathbf{Set}$ (where $1 = \{x\}$) is isomorphic to the category $\mathbf{Set}_*$ of pointed sets (via the functor sending $1 \xrightarrow{f} A$ to $(A, f(x))$). It is also equivalent (but not isomorphic) to the category **Part** of sets and partial functions. The functor $F : \mathbf{Set}_* \to \mathbf{Part}$ sends (A, a) to $A/\{a\}$ and $f : (A, a) \to (B, b)$ to the partial function which agrees with f at $a' \in A$ when $f(a) \neq b$ is is undefined otherwise. In the other direction, $G : \mathbf{Part} \to \mathbf{Set}_*$ sends a set A to $A^+ = A \cup \{A\}$ with A as a basepoint and a partial function $f : A \to B$ to f^+ defined by $f^+(a) = f(a)$ if $a \in A$ and $f(a)$ is defined, otherwise $f(a) = B$. Note that FG is the identity but GF is not. We see this since **Part** contains the object $\varnothing$ which is the only member of its isomorphism class, but in $\mathbf{Set}_*$ each isomorphism class contains many members.

(c) The Categories $\mathbf{fdMod}_k$ and $\mathbf{fdMod}_k^{\mathbf{op}}$ are equivalent via the dual space functor and the natural isomorphism $1_{\mathbf{fdMod}_k} \to (-)^{**}$ on both sides.

(d) $\mathbf{fdMod}_k$ is also equivalent to $\mathbf{Mat}_k$. We have to choose a basis for every finite dimensional vector space and define $F(v) = dim(V)$ and $F(V \xrightarrow{\theta} W)$ to the matrix representing θ with reference to the chosen basis. $G : \mathbf{Mat}_k \to \mathbf{fdMod}_k$ sends n to k^n and a matrix A to $f : x \to Ax$. The composite FG is the identity on $\mathbf{Mat}_k$ (provided we made good choices for the basis).

1.12 Definition. Given a category $\mathcal{C}$, by a skeleton of $\mathcal{C}$ we mean a full subcategory containing exactly one object from each isomorphism class of objects of $\mathcal{C}$. Note that 1.10 implies for any skeleton $\mathcal{C}'$ of $\mathcal{C}$ the inclusion is part of an equivalence. Any equivalence between skeletal categories is bijective on objects and hence in fact and isomorphism.

1.13 Remark. The following statements are each equivalent to the axiom of choice:

1. Any category has a skeleton.
2. Any category is equivalent to any of its skeletons
3. Any two skeletons of a given category are isomorphic.

2 The Yoneda Lemma

2.1 Definition. A category $\mathcal{C}$ is locally small if for any two objects A, B of $\mathcal{C}$, the collection of all morphisms $A \to B$ in $\mathcal{C}$ is a set $\mathcal{C}(A, B)$. If $\mathcal{C}$ is locally small, then the mapping $B \to \mathcal{C}(A, B)$ is a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$. Given a morphism $g : B \to C$ in $\mathcal{C}$, $\mathcal{C}(A, g) : \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ sends $f \in \mathcal{C}(A, B)$ to gf. Similarly $A \to \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{\mathbf{op}} \to \mathbf{Set}$.

2.2 Lemma. *(Yoneda)*

(i) Let $\mathcal{C}$ be a locally small category, $A \in \mathrm{ob}\mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.

(ii) Moreover, the bijection is natural in A and F.

Proof of Lemma 2.2(i). Given $\alpha : \mathcal{C}(A,-) \to F$, we define $\Phi(\alpha) = \alpha_A(1_A) \in FA$. Conversely given $x \in FA$, we define $\Psi(x) : \mathcal{C}(A,-) \to F$ by $\Psi(x)_B(f) = Ff(x)$ for any $B \in \mathrm{ob}\,\mathcal{C}$ and any $f : A \to B$. We need to verify naturality of $\Psi(x)$, that is we need to show that

$$\begin{tikzcd} \mathcal{C}(A,B) \arrow[r, "\Psi(x)_B"] \arrow[d, "\mathcal{C}(A,g)"] & FB \arrow[d, "Fg"'] \\ \mathcal{C}(A,C) \arrow[r, "\Psi(x)_C"'] & FC \end{tikzcd}$$

commutes. But we have

$$(Fg)(\Psi(x)_B(f)) = (Fg)((Ff)(x))$$

and

$$\Psi(x)_C(\mathcal{C}(A,g)(f)) = \Psi(x)_C(gf) = (F(gf)(x)),$$

since F is a functor these two are equal and the diagram commutes. Given x we have

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x.$$

Given any $\alpha : \mathcal{C}(A,-) \to F$, $B \in \mathrm{ob}\,\mathcal{C}$ and $f : A \to B$, we have

$$\alpha_B(f) = \alpha_B(\mathcal{C}(A,f)(1_A)) = (Ff)(\alpha_A(1_A)) = (Ff)(\Phi(\alpha)) = (\Psi\Phi(\alpha))_B(f)$$

by the naturality of α. □

2.3 Corollary. *For a locally small category $\mathcal{C}$, there is a full and faithful functor $Y : \mathcal{C}^{op} \to [\mathcal{C}, \textbf{Set}]$, the Yoneda embedding, sending $A \in \mathrm{ob}\,\mathcal{C}$ to $\mathcal{C}(A,-)$.*

Proof. Put $F = \mathcal{C}(B,-)$ in 2.2(i), then we have a bijections between morphisms $B \to A$ in $\mathcal{C}(B,A)$ and morphisms $\mathcal{C}(A,-) \to \mathcal{C}(B,-)$ in $[\mathcal{C}, \mathbf{Set}]$, which we take to be the effect of Y on morphisms. We need to check that this is functorial: suppose that we are given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathcal{C}$. Then $Y(g)Y(f) : \mathcal{C}(A,-) \to \mathcal{C}(C,-)$ is determined by its effects on $1_A \in \mathcal{C}(A,A)$, but $Y(f)_A$ sends this to $f \in \mathcal{C}(B,A)$ and $Y(g)_A(f) = \mathcal{C}(C,f)g = fg$ and we also have $Y(fg)_A(1_A) = fg$, so $Y(fg) = Y(f)Y(g)$. □

To explain 2.2(ii), suppose for the moment that $\mathcal{C}$ is small, then $[\mathcal{C}, \mathbf{Set}]$ is locally small since a natural transformation $F \to G$ is a set-indexed family of functions $\alpha_A : FA \to GA$. We have a functor $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ sending (A,F) to $F(A)$ and another functor which is the composition

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1_{[\mathcal{C},\mathbf{Set}]}} [\mathcal{C}, \mathbf{Set}] \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C},\mathbf{Set}](-,-)} \mathbf{Set}.$$

2.2(ii) says these two functors are isomorphic. More precisely, it makes the elementary assertations about the equality of two things to which it reduced.

Proof of Lemma 2.2(ii). Naturality in A: Suppose that we are given $f : A \to B$, a functor F and a natural transformation $\alpha : \mathcal{C}(A, -) \to F$. We have to show that

$$\begin{tikzcd}
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A,-), F)} \arrow[r, "\Phi"] \arrow[d, "\alpha \mapsto \alpha \circ Y(f)"] & FA \arrow[d, "Ff"'] \\
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(B,-), F)} \arrow[r, "\Phi"'] & FB
\end{tikzcd}$$

commutes. But we have

$$\begin{aligned}\Phi(\alpha \circ Y(f)) &= \alpha_B(Y(f)(1_B)) = \alpha_B(f) \\ &= \alpha_B(\mathcal{C}(A,f)(1_A)) = (Ff)(\alpha_A(1_A)) = (Ff)\Phi(\alpha),\end{aligned}$$

as needed.

Naturality in F: Suppose that we are given $\theta : F \to G$ and $\alpha : \mathcal{C}(A, -) \to F$, we have to show that

$$\begin{tikzcd}
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A,-), F)} \arrow[r, "\Phi"] \arrow[d, "\alpha \mapsto \theta \circ \alpha"] & FA \arrow[d, "\theta_A"'] \\
{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A,-), G)} \arrow[r, "\Phi"'] & GA
\end{tikzcd}$$

commutes. We have

$$\theta_A \Phi(\alpha) = \theta_A(\alpha_A(1_A)) = \Phi(\theta \circ \alpha),$$

as needed. □

2.4 Definition. We say a functor $F : \mathcal{C} \to \mathbf{Set}$ is representable if it is naturally isomorphic to $\mathcal{C}(A, -)$ for some A. By a representation of F, we mean a pair (A, x) where $A \in \mathcal{C}$ and $x \in FA$ such that $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We call x a universal element of F, it has the property that any $y \in FB$ is of the form $Ff(x)$ for some unique $f : A \to B$.

2.5 Corollary. *Given 2 representations (A, x) and (B, y) of the same functor F, there is a unique isomorphism $f : A \to B$ in $\mathcal{C}$ such that $Ff(x) = y$.*

Proof. Consider the composite $\mathcal{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{(\Psi(x))^{-1}} \mathcal{C}(A, -)$. By 2.3 there is a unique $f : A \to B$ in $\mathcal{C}$ with $Yf = \Psi(x)^{-1}\Psi(y)$ and a unique $g : B \to A$ with $Yg = (Yf)^{-1}$ and $gf = 1_A$, $fg = 1_B$, since Y is faithful. Moreover, the equation $Yf = \Psi(x)^{-1}\Psi(y)$ is equivalent to $\Psi(x)Yf = \Psi(y)$, but these two are equal iff they have the same effect on 1_B, i.e. iff $Ff(x) = y$. □

2.6 Examples.

(a) The forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$ is represented by $(\mathbb{Z}, 1)$ since for any group there is a unique homomorphism $\mathbb{Z} \to G$ sending 1 to x. Similarly, $U : \mathbf{Top} \to \mathbf{Set}$ is represented by $(1, *)$.

(b) The contravariant power set functor $\mathcal{P}^* : \mathbf{Set^{op}} \to \mathbf{Set}$ is represented by $(\{0,1\},\{1\})$, since if $A' \subset A$ there is a unique $\chi_{A'} : A \to \{0,1\}$ such that $(\chi_{A'})^{-1}(1) = A'$.

(c) For a field k, the composite functor $\mathbf{Mod}_k^{\mathbf{op}} \xrightarrow{(-)^*} \mathbf{Mod}_k \xrightarrow{U} \mathbf{Set}$ is representable by $(k, 1_k)$.

(d) Let G be a group and $\mathcal{G}$ be G as a category. The category $[\mathcal{G},\mathbf{Set}]$ is the category of sets with a G action. The unique representable functor $\mathcal{G} \to \mathbf{Set}$ is the Cayley representation of G, i.e. G itself with action via left multiplication. In this case the Yoneda lemma tells us that this is the free G-set on one generator: natural transformations $\eta : \mathcal{G}(*,-) \to F$ in $[\mathcal{G},\mathbf{Set}]$, that is functions $\eta_* : G \to F* = A$ commuting with the G-action, correspond bijectively to elements of A.

(e) Let $\mathcal{C}$ be any locally small category, A,$B \in \mathrm{ob}\,\mathcal{C}$. Consider the functor $F = \mathcal{C}(-,A) \times \mathcal{C}(-,B) : \mathcal{C}^{\mathbf{op}} \to \mathbf{Set}$. What does it mean for this to be representable? A representation consists of some object P together with an element $(p : P \to A, q : P \to B)$ of FP, such that for any C and any $f : C \to A$, $g : C \to B$ there is a unique $h : C \to P$ such that $ph = f$ and $qh = g$.

$$\begin{tikzcd} A & P \arrow[l, "p"'] \arrow[r, "q"] & B \\ & C \arrow[ul, "f"] \arrow[u, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

We can ask whether this exists in any category $\mathcal{C}$, not necessarily locally small. If it does, we call (P,p,q) a categorical product of A and B and normally denote it $(A\times B, \pi_1, \pi_2)$. Note that in $\mathbf{Set}$ the categorical product and cartesian product coincide. In $\mathbf{Gp}$, $\mathbf{Rng}$, $\mathbf{Top}$, etc... this is also true. A coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{\mathbf{op}}$. We usually denote the coproduct of A and B by $A+B$. In $\mathbf{Set}$ $A+B$ is the disjoint union of A and B, the same in $\mathbf{Top}$. But in $\mathbf{Gp}$, the coproduct is the free product $A * B$. In $\mathbf{AbGp}$ the coproduct $A * B$ is isomorphic to $A \times B$ and is usually denoted $\otimes$. In any poset $(P,\leq)$, the product is a greatest lower bound and the coproduct is the least upperbound.

(f) Assume $\mathcal{C}$ is locally small. Suppose that we are given a parallel pair $A \underset{f}{\overset{g}{\rightrightarrows}} B$ in $\mathcal{C}$. Consider the functor F defined by $FC = \{h : C \to A : fh = gh\}$, a subfunctor of $\mathcal{C}(-,A)$. Is this representable? A representation consists of (E,e) whose $e : E \to A$ satisfies $fe = ge$ and any $h : C \to A$ with $fh = gh$ factors uniquely as ek for some $k : C \to E$. Such an e is called an equalizer of f and g. In $\mathbf{Set}$ we take $E = \{a \in A : f(a) = g(a)\}$ and e to be the inclusion. This construction also works in $\mathbf{Gp}$, $\mathbf{Rng}$, $\mathbf{Mod}_k$ and $\mathbf{Top}$. The dual notion is that of the coequalizer, it exists for any pair of morphisms, but the construction is different.

2.7 Definition. We say a morphism $A \xrightarrow{f} B$ is a monomorphism if $fg = fh$ implies $g = h$ for any pair $C \overset{h}{\underset{g}{\rightrightarrows}} A$. Dually, f is an epimorphism if $kf = ef$ implies $k = l$. We say f is a regular monomorphism if it occurs as an equalizer of a pair of maps. (Dually for epimorphism).

In **Set**, monomorphism, regular monomorphism and injection are all synonyms. If $f : A \to B$ is injective form $C = B \times \{0,1\}/\sim$ where $(b,j) \sim (c,k)$ iff $b = c$ and $j = k$ or $b = c = f(a)$ for some $a \in A$. Then the two injections $B \to C$ have equalizer $\{b \in B : b = f(a)\} \cong A$. If f is not injective, then we can find $x, y : 1 \to A$ with $x \neq y$ but $fx = fy$. Similarly, all epimorphisms are regular epimorphisms and are surjective. These don't hold in all categories. They do hold in **Gp**, but not in **Mon** because the inclusion of $\mathbb{N} \to \mathbb{Z}$ is an epimorphism in **Mon**, also monic, but is not a regular monomorphism since an epic equalizer has to be an isomorphism. Similarly in **Top** monic and injective are the same as are epic and surjective. A regular monomorphism is subspace inclusion and a regular epimorphism is a quotient map. There are bijective continuous maps that are not homeomorphisms, We say $\mathcal{C}$ is balanced if every morphism which is both epic and monic in $\mathcal{C}$ is an isomorphism. So **Set**, **Gp** are balanced but **Top** and **Mon** are not.

2.8 Definition. Let $\mathcal{C}$ be a category, $\mathcal{G}$ be a class of objects of $\mathcal{C}$.

(i) We say that $\mathcal{G}$ is a separating family if, given $A \overset{g}{\underset{f}{\rightrightarrows}} B$, with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ such that $fh \neq gh$.

(ii) We say that $\mathcal{G}$ is a detecting family if, given $f : A \to B$ such that every $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely as fh, then f is an isomorphism.

If $\mathcal{C}$ is a locally small category, this translates to:

1. $\mathcal{G}$ is a separating family iff $\{\mathcal{C}(G, -) | G \in \mathcal{G}\}$ are jointly faithful.
2. $\mathcal{G}$ is a detecting family iff $\{\mathcal{C}(G, -) | G \in \mathcal{G}\}$ are jointly isomorphism reflecting.

2.9 Lemma.

(i) Suppose $\mathcal{C}$ has equalizers for all parallel pairs. Then every detecting family of objects in $\mathcal{C}$ is a separating family.

(ii) Suppose $\mathcal{C}$ is balanced, then every separating family is a detecting family.

Proof.

1. Suppose $\mathcal{G}$ is a detecting family and suppose $A \overset{g}{\underset{f}{\rightrightarrows}} B$ is such that every $h : G \to A$ with $G \in \mathcal{G}$ satisfies $fh = gh$. Then every such h factors uniquely through the equalizer $e : E \to A$ of (f, g), so e is an isomorphism. Hence $f = g$.

2. Suppose $\mathcal{G}$ is a separating family and suppose $f : A \to B$ is such that $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f. Then f is epic since if $h, k : B \to C$ satisfy $hg = kf$, then any $g : G \to B$ must satisfy $kg = hg$ so $k = h$. Similarly, if $D \underset{l}{\overset{m}{\rightrightarrows}} A$ satisfy $fl = fm$, then for any $n : G \to D$ we have $fln = fmn$, so ln and mn are both factorizations of fln through f, so they are equal. Hence $l = m$, so f is monic. This f is an isomorphism since $\mathcal{C}$ is balanced.

$\square$

2.10 Examples.

(a) $\mathrm{ob}\,\mathcal{C}$ is always a detecting and separating family for $\mathcal{C}$. Eg. If $f : A \to B$ is such that all $g : G \to B$ factor uniquely through f then there exists $h : B \to A$ such that $fh = 1_B$, then fh and 1_A are both factorizations of f through f, so they are equal.

(b) For any locally small $\mathcal{C}$, $\{YA : A \in \mathrm{ob}\,\mathcal{C}\}$ is a separating and detecting family for $[\mathcal{C}, \mathbf{Set}]$. For if $\alpha : F \to G$ is an arbitrary natural transformation, then if every $YA \to G$ factors uniquely through α, α_A is bijective, and if this holds for all A then α is an isomorphism.

(c) $\{1\}$ is both a separating and detecting family for **Set** since $\mathbf{Set}(1, -)$ is isomorphic to the identity functor. $\{\mathbb{Z}\}$ is both for **Gp** and **AbGp** since $\mathbf{Gp}(\mathbb{Z}, -)$ is isomorphic to the forgetful functor.

(d) $\{1\}$ is a separating family for **Top** since $U : \mathbf{Top} \to \mathbf{Set}$ is faithful. However, there is no detecting set of objects: for any infinite cardinal κ, we can find a set X with cardinality of κ and two topologies τ_1, τ_2 on X such that $\tau_1 \subset \tau_2$, but the two topologies coincide on any subset of X of cardinality less then κ. For ω take the discrete and finite complement topologies. Given any set $\mathcal{G}$ of objects of **Top**, choose $\kappa > \mathrm{card}(UG)$ for all $G \in \mathcal{G}$. Then $\mathcal{G}$ cannot detect the fact that $1_X : (X, \tau_1) \to (X, \tau_2)$ is not an isomorphism.

(e) Let $\mathcal{C}$ be the category of connected, pointed cw-complexes and homotopy classes of continuous maps between them. JHC Whiteheads Theorem asserts that if $f : X \to Y$ in this category induces isomorphisms $\pi_n(X) \to \pi_m(Y)$ for all n, m, then it is an isomorphism by, say, S^n, so this says that $\{S^n : n \geq 1\}$ is a detecting set for $\mathcal{G}$. If $\{G_i | i \in I\}$ where a separating family, then $X \mapsto \prod_{i \in I} \mathcal{C}(G, X)$ would be faithful.

2.11 Definition. Let $\mathcal{C}$ be a category, $P \in \mathrm{ob}\,\mathcal{C}$. We say P is projective if, given any diagram of the form

$$\begin{tikzcd} & P \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with f epic, there exists $h : P \to A$ with $fh = g$. If $\mathcal{C}$ is locally small, this says that $\mathcal{C}(P, -)$ preserves epics. We say that P is injective in $\mathcal{C}$ if it is projective in $\mathcal{C}^{\mathbf{op}}$. More generally, if η is a class of epimorphisms in $\mathcal{C}$, we say that P is η-projective if the above holds true for all $f \in \eta$.

2.12 Lemma. *Let $\mathcal{C}$ be locally small. Then for any $A \in \operatorname{ob}\mathcal{C}$, YA is η-projective in $[\mathcal{C}, \boldsymbol{Set}]$ where η is the class of natural transformations α such that α_B is surjective for all B. In fact these are all epimorphisms.*

Proof.

$$\begin{tikzcd} & YA \arrow[d, "\beta"] \\ F \arrow[r, "\alpha"] & G \end{tikzcd}$$

β corresponds to some $y \in GA$, α_A is surjective, so $y = \alpha_A(x)$ for some x, i.e. there exists $\gamma : YA \to F$ with $\alpha\gamma = \beta$ by the naturality of the Yoneda lemma. To be exact, if Ψ is defined as in the proof of the Yoneda lemma, then

$$\Psi(\beta) = y = \alpha_A(y) = \alpha_A(\Psi(\gamma)) = \Psi(\alpha\gamma),$$

since Ψ is a bijection $\beta = \alpha\gamma$. □

3 Adjunctions

3.1 Definition. Suppose that we are given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We say that F is left adjoint to G if we are given, for each $A \in \operatorname{ob}\mathcal{C}$ and $B \in \operatorname{ob}\mathcal{D}$, a bijection between morphisms $FA \to B$ and $A \to GB$ which is natural in A and B. If $\mathcal{C}$ and $\mathcal{D}$ are locally small, this means that the functors $\mathcal{C}^{\mathbf{op}} \times \mathcal{D} \to \mathbf{Set}$ sending (A, B) to $\mathcal{D}(FA, B)$ and to $\mathcal{C}(A, GB)$ are naturally isomorphic. $F \dashv G$ means F is left adjoint to G, i.e. $\mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)$.

3.2 Examples.

(a) The free functor $F : \mathbf{Set} \to \mathbf{Gp}$ is left adjoint to the forgetful functors $U : \mathbf{Gp} \to \mathbf{Set}$. For any function $f : A \to UG$, there is a unique homomorphism $f : FA \to G$ extending f. Similarly for free rings, R-modules, etc...

(b) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint D sending A to A with the discrete topology. Since $A \to UX$ is continuous as a map from $DA \to X$. U also has a right adjoint, I, sending A to the indiscrete topology.

(c) The functor $\operatorname{ob} : \mathbf{Cat} \to \mathbf{Set}$ has a left adjoint D, sending A to the discrete category where objects are members of A (since a functor $DA \to C$ is uniquely determined by its effects on objects), and a right adjoint I sending A to the preorder with objects $a \in A$ and one morphism $a \to b$ for all $(a, b) \in A \times A$ (again, a functor $C \to IA$ is uniquely determined by its effect on objects). In this case D also has a left adjoint, π_0 sending $\mathcal{C}$ to its set of connected components, i.e. equivalence classes of objects for the equivalence relation $\sim$ generated by $U \sim V$ is there exists a morphism $U \to V$ (once again, a functor $\mathcal{C} \to DA$ is determined by by its effects on objects, but this function $\mathrm{ob} : \mathcal{C} \to A$ must be constant on connected components).

(d) Let 1 denote the category with one object and one morphism. For any $\mathcal{C}$, there is a unique functor $\mathcal{C} \to 1$, a left adjoint, if it exists, picks out an initial object of $\mathcal{C}$, the right adjoint picks out a terminal object.

(e) **Idem** the category whose objects are pairs (A, e) where A is a set and $e : A \to A$ satisfies $e^2 = e$, and morphisms $f : (A, e) \to (A', e')$ are functions such that $e'f = fe$. We have a functor $G : \mathbf{Idem} \to \mathbf{Set}$ sending (A, e) to $\operatorname{im} e$ and a functor $F : \mathbf{Set} \to \mathbf{Idem}$ sending A to $(A, 1_A)$. F is both left and right adjoint to G: a morphism $f : FA \to (B, e)$ must take values in GB, but any function $A \to GB$ is a morphism $FA \to (B, e)$. On the other hand, a morphism $g : (B, e) \to FA$ must satisfy $g(b) = g(e(b))$ for any $b \in B$, so it is uniquely determined by its restriction to GB. Note that $GF = 1_{\mathbf{Set}}$ but $FG \cong 1_{\mathbf{Idem}}$.

(f) Let (X, T) be a topological space. If we think of T as a poset, then $T \to \mathcal{P}X$ is a functor. The operation $A \mapsto \mathring{A}$ gives a right adjoint to this functor, since by definition we have $U \subset A$ iff $U \subset \mathring{A}$ for $U \in T$. Similarly, closure is a left adjoint for the inclusion of the closed sets.

(g) The functor $\mathcal{P}^* : \mathbf{Set} \to \mathbf{Set}^{\mathbf{op}}$ is left adjoint to $\mathcal{P}^* : \mathbf{Set}^{\mathbf{op}} \to \mathbf{Set}$, since morphisms $\mathcal{P}^*A \to B$ in $\mathbf{Set}^{\mathbf{op}}$ are functions $B \to \mathcal{P}^*A$ in $\mathbf{Set}$, which corresponds to relations $B \twoheadrightarrow A$ and morphisms $A \to \mathcal{P}^*B$ in $\mathbf{Set}$ corresponds to relations $A \twoheadrightarrow B$. These correspond bijectively, in a natural way. This becomes symmetric, if we write this as $\mathbf{Set}(A, \mathcal{P}^*B) \cong \mathbf{Set}(B, \mathcal{P}^*A)$. We say $\mathcal{P}^*$ is self adjoint on the right.

(h) Given two sets, A and B and some relations $R \subset A \times B$, we have a mapping $(-)^r : \mathcal{P}A \to \mathcal{P}B$ sending $S \subset A$ to S^r its related elements via R. and $(-)^l : \mathcal{P}B \to \mathcal{P}A$ sending $T \subset A$ to T^l, the elements in A related to T. These are covariant functors, adjoint to each other on the right, since $T \subset S^r$ iff $S \times T \subset R$ iff $S \subset T^l$.

3.3 Theorem. *Suppose that we are given $G : \mathcal{D} \to \mathcal{C}$. For each object A of $\mathcal{C}$, consider the category $(A \downarrow G)$ whose objects are pairs (B, f) with $B \in \mathrm{ob}\,\mathcal{D}$ and $f : A \to GB$ in $\mathcal{C}$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof. Suppose G has a left adjoint F. For any A, the morphism $1_{FA} : FA \to FA$ corresponds to a morphism $\eta_A : A \to GFA$, called the unit of the adjunction. We claim (FA, η_A) is an initial object of $(A \downarrow G)$, For, given an arbitrary (B, f), the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "Gg"] \\ & GB \end{tikzcd}$$

commutes iff $f : A \to GB$ is the morphism corresponding to $FA \xrightarrow{1_{FA}} FA \xrightarrow{g} B$.

Suppose that we are given an initial object of $(A \downarrow G)$ for each $A \in \mathrm{ob}\,\mathcal{C}$. Denote this object by (FA, η_A): this defines F on objects. given $f : A \to A'$ in $\mathcal{C}$, define $Ff : FA \to FA$ to be the unique morphism such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"'] \\ A' \arrow[r, "\eta_{A'}"'] & GFA' \end{tikzcd}$$

commutes, i.e. the unique morphism $(FA, \eta_A) \to (FA', \eta_{a'} f)$ in $(A \downarrow G)$.

If we have $f' : A' \to A''$, then $F(f'f)$ and $(Ff)(Ff')$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$, so they must be equal. Hence F is a functor and η is a natural transformation $1_{\mathcal{C}} \to GF$. We have a bijective correspondence between morphisms $f : A \to GB$ and morphisms $g : FA \to B$. Take g to be the unique morphism such that $(Gg)\eta_A = f$. Naturality in B is immediate from the form of the definition. Naturality in A follows from the fact that η is a natural transformation. $\square$

3.4 Corollary. *Any two left adjoints F, F' for a given functor G are (canonically) isomorphic.*

Proof. For each A, there is a unique isomorphism $(FA, \eta_A) \to (F'A, \eta'_A)$ in $(A \downarrow G)$ called $h_A : FA \to F'A$. To verify that this is natural in A we first need to establish that if Ψ is the mapping between $\mathcal{C}(A, GB)$ and $\mathcal{D}(FA, B)$, then if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & GB \arrow[d, "Gh"] \\ & GC \end{tikzcd}$$

commutes then so does

$$\begin{tikzcd} FA \arrow[r, "\Psi_B(f)"] \arrow[dr, "\Psi_C(g)"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

that is, given $(Gh)f = g$, we want $h\Psi_B(f) = \Psi_C(g)$. But this follows from the fact that Ψ can be seen as a natural transformation between $\mathcal{C}(A, G-)$ and $\mathcal{D}(FA, -)$. That is, we have, for any $h : B \to C$, $\mathcal{D}(FA, h) \circ \Psi_B = \Psi_C \circ \mathcal{C}(A, Gh)$, so we have $h\Psi_B(f) = \Psi_C((Gh)f) = \Psi_C(g)$ as needed. Now, it is clear that all the triangles in this diagram commute for any $f : A \to A'$:

$$\begin{tikzcd}
GFA \arrow[rr, "Gh_A"] \arrow[dd, "GFf"'] & & GF'A \arrow[dd, "GF'f"] \\
& A \arrow[ul, "\eta_A"'] \arrow[ur, "\eta'_A"] \arrow[dl, "\eta_{A'}f"] \arrow[dr, "\eta'_{A'}f"'] & \\
GFA' \arrow[rr, "Gh_{A'}"'] & & GF'A'
\end{tikzcd}$$

Thus they commute when we pass through Ψ as above, so the square

$$\begin{tikzcd}
FA \arrow[r, "h_A"] \arrow[d, "Ff"'] & F'A \arrow[d, "F'f"] \\
FA' \arrow[r, "h_{A'}"'] & F'A'
\end{tikzcd}$$

commutes. Hence the isomorphism is natural. $\square$

3.5 Lemma. *Given functors* $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D} \underset{H}{\overset{K}{\leftrightarrows}} \mathcal{E}$ *with* $F \dashv G$ *and* $H \dashv K$ *then* $HF \dashv GK$.

Proof. We have bijections between morphisms $HFA \to C$ and morphisms $FA \to KC$ and morphisms $A \to GKC$, natural in A and C. $\square$

3.6 Corollary. *Suppose that we are given a commutative square of categories and functors*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G_1"] \arrow[d, "G_2"'] & \mathcal{D} \arrow[d, "G_3"] \\
\mathcal{E} \arrow[r, "G_4"'] & \mathcal{F}
\end{tikzcd}$$

and suppose that each G_i *has a left adjoint* F_i. *Then*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "F_4"] \arrow[d, "F_3"'] & \mathcal{E} \arrow[d, "F_2"] \\
\mathcal{D} \arrow[r, "F_1"'] & \mathcal{C}
\end{tikzcd}$$

commutes up to isomorphism.

Proof. Immediate from 3.4, 3.5 □

Given an adjunction $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$, $F \dashv G$, we have a natural transformation $1_{\mathcal{C}} \to GF$ and dually a natural transformation $FG \to A_{\mathcal{D}}$, the unit and counit of the adjunction.

3.7 Theorem. *Given functors $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ specifying an adjunction $F \dashv G$ is equivalent to specifying a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ satisfying the triangular identities:*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Proof. Suppose that we are given an adjunction $F \dashv G$ with unit η and counit ϵ. By definition $\eta_A : A \to GFA$ corresponds to $1_{FA} : FA \to FA$ and $\epsilon_{FA} : FGFA \to FA$ corresponds to $1_{GFA} : GFA \to GFA$ so $\epsilon_{FA}F\eta_A : FA \to FA$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$. Hence $\epsilon_{FA}F\eta_A = 1_{FA}$, dually for the other triangle.

Conversely, suppose that we are given η and ϵ satisfying the identities. Given $f : A \to GB$, define $\Phi(f) : FA \to B$ to be the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$. Given $g : FA \to B$, define $\Psi(g) : A \to GB$ to be $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. As in the proof of 3.3, we know that Φ and Ψ are natural in A and B. To show that they are inverse to each other:

$$\begin{aligned} \Psi\Phi(f) = A \xrightarrow{\eta_A} GFA \xrightarrow{G\Phi(f)} GB &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB = A \xrightarrow{f} GB \end{aligned}$$

by the naturality of η and the second triangular identity. Similarly $\Phi\Psi(g) = g$ for any $g : FA \to B$. □

3.8 Lemma. *Suppose that we are given $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ with $F \dashv G$ and counit $\epsilon : FG \to 1_{\mathcal{D}}$. Then*

(i) G is faithful iff ϵ_B is an epimorphism for all B.

(ii) G is full and faithful iff ϵ is an isomorphism.

Proof.

1. Suppose ϵ_B is epic for all B, and suppose $g, g' : B \to B'$ satisfying $Gg = Gg'$. Then the morphisms $FGB \rightrightarrows B'$ corresponding to Gg and Gg' are equal, but these are $g\epsilon_B$ and $g'\epsilon_B$ respectively and e_B is epic, so $g = g'$. Conversely, suppose G is faithful, and $B \underset{g}{\overset{g'}{\rightrightarrows}} B'$ satisfy $g\epsilon_B = g'\epsilon_B$, then $Gg = Gg'$ so $g = g'$.

2. Suppose ϵ is an isomorphism. By i we know G is faithful, so we need only prove G is full. Suppose that we are given $g : GB \to GB'$ transposing we get $\bar{f} : FGB \to B'$. Then if we set $g = \bar{f}(\epsilon_B^{-1}) : B \to B'$ we have Gg corresponding to $\bar{f}$. so $Gg = f$.

   Conversely, suppose G is full and faithful. Then $GB \xrightarrow{\eta_{GB}} GFGB$ must be of the form Gh for some $h : B \to FGB$. But $(G\epsilon_B)(n_{GB}) = 1_{GB}$ so $\epsilon_B h = 1_B$ since G is faithful. $h\epsilon_B$ corresponds, under the adjunction, to $GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \xrightarrow{Gh} GFGB$ and $Gh = 1_{GB}$ so $h\epsilon_B = 1_{FGB}$.

□

3.9 Definition. By a reflection we mean an adjunction satisfying the conditions of 3.8(ii). We say that $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$ if the inclusion is full and faithful and has a left adjoint.

3.10 Examples.

(a) The subcategory **AbGp** of **Gp** is reflective. Given an arbitrary group G, let G' be the derived subgroup, then G/G' is abelian and any morphism $G \to A$ where A is abelian factors uniquely through $G \to G/G'$.

(b) The subcategory **tfAbGp**, of torsion free abelian groups is reflective in **AbGp**. The reflector sends $A \to A/\operatorname{tor} A$, the torsion free subgroup. Also, the subcategory **tAbGp** of torsion abelian groups is reflective in **AbGp**. The counit of this adjunction is the inclusion $A_t \hookrightarrow A$.

(c) The category **kHaus** of compact hausdorf spaces is reflective in **Top**. The reflector is the Stone-Čech compactification $x \mapsto \beta x$.

3.11 Lemma. *Suppose we have an equivalence of categories* $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$, $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. *There exist natural transformations* $\alpha' : 1_{\mathcal{C}} \to GF$, $\beta' : FG \to 1_{\mathcal{D}}$, *which satisfy the triangular identities so that* $F \dashv G$ *and also* $G \dashv F$.

Proof. First note that

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "\alpha"] \arrow[d, "\alpha"] & GF \arrow[d, "GF\alpha"'] \\ GF \arrow[r, "\alpha_{GF}"'] & GFGF \end{tikzcd}$$

commutes by the naturality of α. But α is pointwise epic, so $GF\alpha = \alpha_{GF}$. Similarly $FG\beta = \beta_{FG}$. Now define $\alpha' = \alpha$ and let β' be the composite

$$FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}.$$

To verify the triangular identities, the composite $(G\beta')(\alpha'_G)$ is

$$\begin{aligned} G &\xrightarrow{\alpha_G} GFG \xrightarrow{(GFG\beta')^{-1}} GFGFG \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{(\alpha_{GFG})^{-1}} GFG \xrightarrow{G\beta} G = G \xrightarrow{1_G} G \end{aligned}$$

by the naturality of α and $GF\alpha = \alpha_{GF}$. Similarly $(\beta'_F)(F\alpha)$ is

$$\begin{aligned} F &\xrightarrow{F\alpha} FGF \xrightarrow{(\beta_{FAF})^{-1}} FGFGF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{(\beta_F)^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{(FGF\alpha)^{-1}} FGF \xrightarrow{\beta_F} F = F \xrightarrow{1_F} F \end{aligned}$$

by naturality of β and $\alpha_{GF} = GF\alpha$. $\square$

4 Limits

4.1 Definition. Let J be a category (almost always small, often finite). By a diagram of shape J in a category $\mathcal{C}$ we mean a functor $D : J \to \mathcal{C}$. The objects $D(j)$, $j \in \mathrm{ob}\, J$ are called verticies of D and the morphisms $D(\alpha)$, $\alpha \in \mathrm{mor}\, J$ are the edges of D. For example, if J is the finite category

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

a diagram of shape J is a commutative square. If J is the category

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd},$$

a diagram of shape J is a not necessarily commutative square.

For any object A of $\mathcal{C}$ and any J, we have a constant diagram $\triangle A$ of shape J, all of whose verticies are A and all of whose edges are 1_A. By a cone over $D : J \to \mathcal{C}$ with a summit of A, we mean a natural transformation $\lambda : \triangle A \to D$. Equivalently this is a family $(\lambda_j : A \to D(j) : j \in \mathrm{ob}\, J)$ of morphisms (the legs of the cone), such that

$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"] & & D(j') \end{tikzcd}$$

commutes for every $\alpha : j \to j'$ in J. Note that $\triangle$ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and a cone over J is an object of the arrow category $(\triangle \downarrow D)$. We say a cone $(L \xrightarrow{\lambda_j} D(j) : j \in \mathrm{ob}\, J)$ is a limit for D if it is a terminal object of $(\triangle \downarrow D)$.

4.2 Definition. We say that $\mathcal{C}$ has limits of shape J if $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right adjoint. By 3.3 this is equivalent to saying that every diagram $D : J \to \mathcal{C}$ has a limit.

4.3 Examples.

(a) If $J = \varnothing$, then $[J, \mathcal{C}]$ has a unique object and the set of cones over it is isomorphic to $\mathcal{C}$. So a limit of this diagram is a terminal object of $\mathcal{C}$ and a colimit is an initial object.

(b) If J is a discrete category, a diagram of shape J is just a family of objects of $\mathcal{C}$ and a cone over it is a family of morphisms $(A \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$. So a limit is a product $\prod_{j \in \operatorname{ob} J} D(j)$. Similarly the colimit produces the coproduct.

(c) Let J be the finite category $\cdot \rightrightarrows \cdot$ (so a diagram of shape J is a parallel pair of morphisms). A cone over such a diagram is of the form

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[dr, "k"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

so $fh = k = gh$ or equivalently a morphism $C \xrightarrow{h} A$ satisfying $fh = gh$. The limit for this diagram is an equalizer for (f, g) (and a colimit is a coequalizer).

(d) Let J be the finite category

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

then a diagram of shape J is a pair of morphisms

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

with common codomain. A cone over this has the form

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, "l"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

satisfying $hf = l = gk$ or alternatively, a completion of the diagram into a commutative square. A terminal such completion is called a pullback for the pair (f, g). If $\mathcal{C}$ has products and equalizers, then it has pullbacks. Form the product

$$\begin{tikzcd} A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \\ B & \end{tikzcd}$$

and then the equalizer $E \xrightarrow{e} A \times B \overset{f\pi_1}{\underset{g\pi_2}{\rightrightarrows}} C$. Then

$$\begin{tikzcd} E \arrow[r, "f\pi_1"] \arrow[d, "g\pi_2"] & A \\ B & \end{tikzcd}$$

is a pullback of f along g. A colimit of the shape $J^{\mathbf{op}}$ is called a pushout of (f, g).

4.4 Theorem.

(i) If $\mathcal{C}$ has equalizers and all small (respectively finite) products, then $\mathcal{C}$ has all small (respectively finite) limits.

(ii) If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Proof.

1. Let J be small (respectively finite) and $D : J \to \mathcal{C}$ a diagram. Form the products $P = \prod_{j \in \operatorname{ob} J} D(j)$ and $Q = \prod_{\alpha \in \operatorname{mor} J} D(\operatorname{cod} \alpha)$. Now form $P \overset{g}{\underset{f}{\rightrightarrows}} Q$ defined by $\pi_\alpha f = \pi_{\operatorname{cod} \alpha}$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\operatorname{dom} \alpha}$ and the equalizer $E \xrightarrow{e} P$ of (f, g). We claim $(E \xrightarrow{\pi_j e} D(j) : j \in \operatorname{ob} J)$ is a limit cone for D. It is a cone since for any edge $\alpha : j \to j'$, we have $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e$. Given any cone $(A \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$ we get a unique $\lambda : A \to P$ such that $\pi_j \lambda = \lambda_j$, but then
$$\pi_\alpha f \lambda = \pi_j \lambda = \lambda_j = D(\alpha)\lambda_i = D(\alpha)\pi_i \lambda = \pi_\alpha g \lambda$$
for all $\alpha : i \to j$, so $f\lambda = g\lambda$, so λ factors uniquely as $e\mu$, so μ is the unique factorization of $(\lambda_j : j \in \operatorname{ob} J)$ through $(\pi_j e : j \in \operatorname{ob} J)$.

2. We can construct the product $A \times B$ as the pullback of
$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$
where 1 is the terminal object and then construct
$$\prod_{i=1}^{n} A_i = (...((A_1 \times A_2) \times A_3)...) \times A_n.$$
Then we can form the equalizer of $A \overset{g}{\underset{f}{\rightrightarrows}} B$ as the pullback of
$$\begin{tikzcd} & B \arrow[d, "(1_B, 1_B)"] \\ A \arrow[r, "(f,g)"] & B \times B \end{tikzcd},$$

since a cone over this diagram consists of

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "h"] & B \\ A & \end{tikzcd}$$

satisfying $fh = 1_B k$ and $gh = 1_B k$.

$\square$

4.5 Definition. let $F : \mathcal{C} \to \mathcal{D}$ be a functor, J a (small) category.

(i) We say F preserves limits of shape J, if, given $D : J \to \mathcal{C}$ and a limit cone $(L \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$, then $(FL \xrightarrow{F\lambda_j} FD(j) : j \in \operatorname{ob} J)$, is a limit cone for FD in $\mathcal{D}$.

(ii) We say F reflects limits of shape J, if, given $D : J \to \mathcal{C}$ and a cone $(L \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$ such that $(FL \xrightarrow{F\lambda_j} FD(j) : j \in \operatorname{ob} J)$ is a limit cone for FD, then the original is a limit for D.

(iii) We say F creates limits if given a limit $(M \xrightarrow{\mu_j} FD(j) : j \in \operatorname{ob} J)$ for FD, then there exists a limit cone $(L \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$ over D mapping to a limit for FD.

4.6 Corollary. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. In any version of 4.4, we may replace '$\mathcal{C}$ has' by either '$\mathcal{C}$ has and F preserves' or '$\mathcal{C}$ has and F creates'.*

4.7 Examples.

(a) $U : \mathbf{Gp} \to \mathbf{Set}$ creates all small limits, but does not preserve or reflect colimits (since coproducts in **Gp** are larger then in **Set**).

(b) $U : \mathbf{Top} \to \mathbf{Set}$ preserves all small limits and colimits, but does not reflect them.

(c) $U : \mathcal{C}/B \to \mathcal{C}$ creates colimits, since a diagram $D : J \to \mathcal{C}/B$ is the same thing as a diagram $UD : J \to \mathcal{C}$ together with a cone $(UD(j) \xrightarrow{D(j)} B : j \in \operatorname{ob} J)$. So given a colimit $(UD(j) \xrightarrow{\lambda_j} L : j \in \operatorname{ob} J)$ in $\mathcal{C}$, we get $h : L \to B$ such that the λ_j are all morphisms $D(j) \to h$ in $\mathcal{C}/B$. They form a cones under D and it is a colimit. But $U : \mathcal{C}/B \to \mathcal{C}$ does not preserve or reflect products: the product of $f : A \to B$ and $g : C \to B$ is $\mathcal{C}/B$ is the diagonal of the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

in $\mathcal{C}$.

(d) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\mathrm{ob}\,\mathcal{C}}$ creates all limits and colimits which exist in $\mathcal{D}$. To prove this, let $D : J \to [\mathcal{C}, \mathcal{D}]$ be a diagram, we can consider it as a functor $J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \mathrm{ob}\,\mathcal{C}$ we can form a limit cone $(LA \xrightarrow{\lambda_{j,A}} D(j, A) : j \in \mathrm{ob}\, J)$ for $D(-, A) : J \to \mathcal{D}$. For each $f : A \to B$, the composites $(LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B) : j \in \mathrm{ob}\, J)$ form a cone over $D(-, B)$. So they induce a unique $Lf : LA \to LB$ such that $\lambda_{j,B} Lf = D(j, f)\lambda_{j,A}$.

Given $g : B \to C$, we need to show that $L(gf)$ and $(Lg)(Lf)$ are equal. But,

$$\lambda_{j,C}(Lg)(Lf) = (D(j, g))(D(j, f))\lambda_{j,A} = (D(j, gf))\lambda_{j,A} = \lambda_{j,C}(L(gf)),$$

that is they are both factorizations of the same cone $(LA \xrightarrow{(D(j,gf))\lambda_{j,A}} D(j, C) : j \in \mathrm{ob}\, J)$ over $D(-, C)$ through the limit LC, so they are equal by uniqueness. Hence L is a functor $L : \mathcal{C} \to \mathcal{D}$ and each $\lambda_{j,-}$ is a natural transformation $L \to D(j, -)$. The $(\lambda_{j,-} : j \in \mathrm{ob}\, J)$ also form a cone over D (regarded as a diagram of shape J in $[\mathcal{C}, \mathcal{D}]$) with summit L. Ex: verify that this is indeed a limit cone in $[\mathcal{C}, \mathcal{D}]$.

(e) The inclusion functor **AbGp** $\to$ **Gp** reflects coproducts, but does not preserve them. A free product (coproduct) $G * H$ is never abelian unless G or H is trivial, and in that event it is also a coproduct in **AbGp**.

4.8 Remark. A morphism $f : A \to B$ is a monomorphism iff

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback. Hence any functor that preserves/reflects pullbacks, also preserves/reflects monomorphisms.

4.9 Theorem. *Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Then G preserves limits that exist in $\mathcal{D}$.*

Proof 1. Suppose $\mathcal{C}$ and $\mathcal{D}$ both have limits of some shape J. Then the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\triangle"'] & \mathcal{D} \arrow[d, "\triangle"] \\ {[J, \mathcal{C}]} \arrow[r, "{[J,F]}"] & {[J\mathcal{D}]} \end{tikzcd}$$

commutes and all functors have right adjoints so by Corollary 3.6,

$$\begin{tikzcd} {[J, \mathcal{D}]} \arrow[r, "{[J,G]}"] \arrow[d, "\lim"] & {[J, \mathcal{C}]} \arrow[d, "\lim"] \\ \mathcal{D} \arrow[r, "G"] & \mathcal{C} \end{tikzcd}$$

commutes upto isomorphism. But this says that G preserves limits of shape J. $\square$

Proof 2. Let $D : J \to \mathcal{D}$ and $(L \xrightarrow{\lambda_j} D(j) : j \in \operatorname{ob} J)$ be a limit for D. Given a cone $(A \xrightarrow{\mu_j} GD(j) : j \in \operatorname{ob} J)$ over GD in $\mathcal{C}$, we get a family of morphisms $(FA \xrightarrow{\bar{\mu}_j} D(j) : j \in \operatorname{ob} J)$, which forms a cone over D by the naturality of $\mu \mapsto \bar{\mu}$. That is we have a natural isomorphism $\alpha : \mathcal{C}(A, G-) \to \mathcal{D}(FA, -)$ and $\beta : i \to j$ giving diagram

$$\begin{tikzcd}
\mathcal{C}(A, GD(i)) \arrow[r, "\alpha_{D(i)}"] \arrow[d, "\mathcal{C}(A,GD(\beta))"'] & \mathcal{D}(FA, D(i)) \arrow[d, "\mathcal{D}(FA,D(\beta))"] \\
\mathcal{C}(A, DG(j)) \arrow[r, "\alpha_{D(j)}"'] & \mathcal{D}(FA, D(j))
\end{tikzcd}$$

Now, $\bar{\mu}_i = \alpha_{D(i)}(\mu_i)$, and since $GD(\beta)\mu_i = \mu_j$ because they form a cone, we see that μ_i maps to $\bar{\mu}_j$ through $\alpha_{D(j)}\mathcal{C}(A, GD(\beta))$. Since the diagram commutes, this means that $D(\beta)\bar{\mu}_i = \bar{\mu}_j$. So we get a unique $\bar{\mu} : FA \to L$ such that $\lambda_j \bar{\mu} = \bar{\mu}_j$ for each j. i.e a unique $\mu : A \to GL$ such that $(G\lambda_j)\mu = \mu_j$. So the $G\lambda_j$ forms a limit cone. $\square$

Our aim is to prove that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves 'all' limits, then G has a left adjoint.

4.10 Lemma. *Suppose that $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then $(A \downarrow G)$ has limits of shape J for each $A \in \operatorname{ob}\mathcal{C}$ and $U : (A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose that we are given $D : J \to (A \downarrow G)$. Write $D(j)$ as $(UD(j), A \xrightarrow{f_j} GUD(j))$, then the f_j form a cone over $GUD : J \to \mathcal{C}$. So if $(L \xrightarrow{\lambda_j} UD(j) : j \in \operatorname{ob} J)$ is a limit for UD, then we get a unique $f : A \to GL$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f_j"'] & GL \arrow[d, "G\lambda_j"] \\
& GUD(j)
\end{tikzcd}$$

commutes for each j. i.e. such that each λ_j form a cone over D with summit (L, f), since they form a cone over UD and U is faithful. Given any cone $((B, g) \xrightarrow{\mu_j} (UD(j), f_j) : j \in \operatorname{ob} J)$ over D in $(A \downarrow G)$ the μ_j also form a cone over UD with sumit B, so they induce a unique $\mu : B \to L$ such that $\lambda_j \mu = \mu_j$ for all j. We need to show $(G\mu)g = f$, but there are factorizations of the same cone over GUD through GL so they are equal. So $\mu : (B, g) \to (L, f)$ in $(A \downarrow G)$ and its the unique factorization of $(\mu_j : j \in \operatorname{ob} J)$ through $(\lambda_j : j \in \operatorname{ob} J)$ in this category. $\square$

4.11 Lemma. *Specifying an initial object for a category $\mathcal{C}$ is equivalent to specifying a limit for $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.*

Proof. If I is an initial object, the unique isomorphisms $(I \to A : A \in \mathrm{ob}\,\mathcal{C})$ form a cone over $1_{\mathcal{C}}$. Given any cone $(S \xrightarrow{\lambda_A} A : A \in \mathrm{ob}\,\mathcal{C})$ over $1_{\mathcal{C}}$, $\lambda_I : S \to I$ is a factorization through the cone with sumit I, since

$$\begin{tikzcd} S \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d] \\ & A \end{tikzcd}$$

commutes.

Suppose that we are given a limit cone $(L \xrightarrow{\lambda_A} A : A \in \mathrm{ob}\,\mathcal{C})$ for $1_{\mathcal{C}}$. We need to show that, for each A, λ_A is the unique morphism $L \to A$. Given $f : L \to A$, we have a commutative triangle

$$\begin{tikzcd} L \arrow[r, "\lambda_L"] \arrow[dr, "\lambda_A"'] & L \arrow[d, "f"] \\ & A \end{tikzcd}$$

In particular, $\lambda_A \lambda_L = \lambda_A$ for all A, so λ_L is a factorization of the limit cone through itself. So $\lambda_L = 1_L$, and hence any $f : L \to A$ satisfies $f = \lambda_A$. □

4.12 Theorem. *(Primitive Adjoint Functor Theorem) If $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.*

Proof. By 4.10 each $(A \downarrow G)$ has all limits. By 4.11 each $(A \downarrow G)$ as an initial object. By 3.3 G has a left adjoint. □

We have a problem, if $\mathcal{D}$ has limits for all diagrams 'as big as itself', then it is a preorder (c.f. example sheet 2 question 4). We call a category complete if it has all small limits

4.13 Theorem. *(General Adjoint Functor Theorem) Let $\mathcal{D}$ be locally small and complete and $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then G has a left adjoint iff G preserves all small limits and satisfies the 'solution set condition': for every $A \in \mathrm{ob}\,\mathcal{C}$, there exists a set of morphisms $\{f_i : A \to GB_i : i \in I\}$ such that every $A \to GB$ factors as $A \xrightarrow{f_i} GB_i \xrightarrow{Gh} GB$ with $h : B_i \to B$ in $\mathcal{D}$.*

Proof. G preserves limits by 4.9, and $\{\eta_A : A \to GFA\}$ is a solution set for A by 3.3.

Each $(A \downarrow G)$ is complete by 4.10, and it inherits local smallness from $\mathcal{D}$. It also satisfies the solution set condition on objects: there is a set of objects $\{C_i : i \in I\}$ such that for any object $C \in (A \downarrow G)$, we have an morphism $C_i \to C$ for some $i \in I$. So we need to show that if $\mathcal{A}$ is complete, locally small and has a solution set of objects, then it has an initial object. Let $\{C_i : i \in I\}$ be a solution set, and form $P = \prod_{i \in I} C_i$. Take $e : E \to P$ the limit of the diagram $P \to P$ with one edge for each morphism $P \to P$ in $\mathcal{A}$. For every object D, we have

$P \to C_i \to D$ for some C_i and hence $E \to P \to D$. Suppose we have $E \overset{g}{\underset{f}{\rightrightarrows}} D$, form their equalizer $h : F \to E$. There exists some $k : P \to F$ and ehk is an endomorphism of P, so $ehke = 1_P e$, e is a monomorphism, so $hke = 1_E$. In particular, h is epic, so $f = g$. $\square$

4.14 Lemma. *Suppose that we are given a pullback square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"'] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

with h monic, then g is monic.

Proof. Suppose $E \overset{m}{\underset{l}{\rightrightarrows}} A$ satisfy $gl = gm$, then $hfl = kgl = kgm = hfm$, but h is monic, so $fl = fm$, so l and m are factorizations of the same cone through a limit, so $l = m$. $\square$

A subobject of A in a category is a monomorphism $A' \to A$. We say that a category is well powered if, for every $A \in \mathrm{ob}\,\mathcal{C}$, there exists a set of subobjects $\{A_i \to A : i \in I\}$ such every subobject $A' \to A$ is isomorphic (in $\mathcal{C}/A$) to some $A_i \to A$. For example **Set**, **Gp** and **Top** are all well powered.

4.15 Theorem. *(Special Adjoint Functor Theorem) Suppose $\mathcal{C}$ is locally small and $\mathcal{D}$ is locally small, completed, well powered and has a coseparating set of objects. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint iff G preserves all small limits.*

Proof. The forward implication follows from 4.9.

We first show that each $(A \downarrow G)$ is complete, locally small, well powered and has a coseparating set. Completeness and local smallness as before. For well poweredness, note that a morphism $h : (B', f') \to (B, f)$ is $(A \downarrow G)$ is monic if it is monic in $\mathcal{D}$ by 4.8. So subobjects of (B, f) in $(A \downarrow G)$ correspond to subobjects $m : B' \to B$ such that f uniquely factors through G in $GB' \to GB$, that is each subobject in $\mathcal{D}$ gives rise to one in $(A \downarrow G)$, . So, up to isomorphism, these form a set. For the coseparating set, let $\{S_i : i \in I\}$ be a coseparating set of $\mathcal{D}$. Then the set $\{(S_i, f) : i \in I, f \in \mathcal{C}(A, GS_i)\}$ is a coseparating set for $(A \downarrow G)$ since if we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f_i"] \arrow[ddr, "(Gl)f'"'] & GB \arrow[d, shift right, "Gh"'] \arrow[d, shift left, "Gk"] \\ & GB' \arrow[d, "l"] \\ & GS_i \end{tikzcd}$$

with $h \neq k$, there exists some $l : B \to S$ with $lh \neq lk$ and l is a morphism $(B', f') \to (S_i, (Gl)f')$ in $(A \downarrow G)$.

We need to show that if $\mathcal{A}$ is complete, locally small, well powered and has a coseparating set $\{S_i : i \in I\}$ of objects, then it has an initial object. First form $P = \prod_{i \in I} A_i$. Let $\{P_i \to P : j \in J\}$ be a representative set of subobjects of P, and form the limit of

$$\begin{tikzcd} P_i \arrow[dr] & \\ \vdots & P \\ P_j \arrow[ur] & \end{tikzcd}$$

whose edges are all the $P_i \to P$ for $j \in J$. If L is the summit of the limit cone, then $L \to P$ is monic by the argument of 4.14 and it is the smallest subobject of P, since it factors through every $P_j \to P$. We claim that L is an initial object of $\mathcal{A}$. Suppose we had two morphisms $L \underset{f}{\overset{g}{\rightrightarrows}} A$, then we could form their equalizer $E \to L$, but $E \to L \to P$ is monic, so $L \to P$ factors through it and hence 1_L factors through $E \to L$, so $E \to L$ is epic and $f = g$.

For existence given $A \in \operatorname{ob} \mathcal{A}$, consider $K = \{(i, f) : i \in I, f : A \to S_i\}$ and form $Q = \prod_{(i,f) \in K} S_i$. We have a canonical map $h : A \to Q$ defined by $\pi_{(i,f)} h = f$ and h is monic since the S_i form a coseparating family. We also have $k : P \to Q$ defined by $\pi_{(i,f)} k = \pi_i$. Form the following pullback

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "l"'] & P \arrow[d, "k"] \\ A \arrow[r, "n"'] & {} \end{tikzcd}$$

then m is monic by 4.14, so $L \to P$ factors through it. So we have a morphism $L \to B \xrightarrow{l} A$. □

4.16 Examples.

(a) If we did not know how to construct free groups, we could use the GAFT to construct a left adjoint for $U : \mathbf{Gp} \to \mathbf{Set}$. We already know that $\mathbf{Gp}$ has and U preserves all small limits (4.7a). So all we need to verify is the solution set condition. Given a set A, any function $A \xrightarrow{f} UG$ factors through $A \to UG' \to UG$, where G' is the subgroup generated by $\{f(a) : a \in A\}$. If $|A| = \kappa$, then $|G'| = \aleph_0 \kappa$, so if we take a set of this size and equip all subsets of it with all possible group structures, plus all possible maps from A, we obtain a solution set.

(b) Consider the category **CLat** of complete latices and $U : \mathbf{CLat} \to \mathbf{Set}$ the forgetful functor. Just as for groups **CLat** has a U preserves all small limits and **CLat** is locally small. However AW Hales showed that, for any cardinal κ, there exists a 3-generator complete lattice of size κ. So the solution set fails for $A = \{1, 2, 3\}$ and U does not have a left adjoint.

(c) Consider the inclusion $I : \mathbf{kHaus} \to \mathbf{Top}$. **kHaus** has small products and F preserves them (Tychonoff). It has equalizers because, given a pair of maps $X \underset{f}{\overset{g}{\rightrightarrows}} Y$, with Y haussdorf, their equalizer is a closed subspace of X and hence compact if X is. **kHaus** and **Top** are locally small. **kHaus** is well-powered since the subobjects of X are all isomorphic to closed subspaces of X. By Uryson's lemma, the closed unit interval $[0, 1]$ is a coseparater for **kHaus**. So by 4.15, I has a left adjoint β, the Stone-Čech compactification functor. Čech's original (1937) construction of βx was as follows: form $P = \prod_{f:X\to[0,1]}[0, 1]$ and then form the closed of the image of the canonical map $X \to P$ (It may be noted that this inspired the proof of the $SAFT$ and possibly the conditions as well).

5 Monads

Suppose that we are given an adjunction $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ $F \dashv G$. What is the 'trace' of this adjunction on the category $\mathcal{C}$? We have the functor $T = GF : \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\eta : 1_{\mathcal{C}} \to T$ and $\mu = G\epsilon_F : TT \to T$. From the triangular identities we get the triangles

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd} \quad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

And from the naturality we get the commutativity of

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd}$$

5.1 Definition. By a monad, $\mathbb{T} = (T, \eta, \mu)$ we mean a functor $T : \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\eta : 1_{\mathcal{C}} \to T$ and $\mu : TT \to T$ satisfying the above 3 diagrams. Any adjunction $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ $F \dashv G$ induces a monad $(GF, \eta, G\epsilon_F)$ on $\mathcal{C}$ and a comonad $(FG, \epsilon, F\eta_G)$ on $\mathcal{D}$.

Given a monad M, the functor $M \times (-) : \mathbf{Set} \to \mathbf{Set}$ has a monad structure with unit $\eta : A \to M \times A$ sending a to (e, a) and multiplication $\mu_A : M \times M \times A \to$

$M \times A$ sending (m, n, a) to (mn, A). This monad is induced by an adjunction, $\mathbf{Set} \underset{F}{\overset{G}{\leftrightarrows}} M\text{-}\mathbf{Set}$ where M-**Set** is the category of sets with an M action, G is the forgetful functor and $FA = M \times A$ (with M action my multiplication on the left factor).

5.2 Definition. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. By a $\mathbb{T}$ algebra, we mean a pair (A, α) where $A \in \mathrm{ob}\,\mathcal{C}$ and $\alpha : TA \to A$ satisfies

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[r, "A"'] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \quad \begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

A morphism $f : (A, \alpha) \to (B, \beta)$ of $\mathbb{T}$ algebras is a morphism $f : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$ algebras and homomorphisms between them, this is also called the Eilenberg-Moore category of $\mathbb{T}$. There is an obvious forgetful functor $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, sending (A, α) to α and f to f.

5.3 Lemma. *$G^{\mathbb{T}}$ has a left adjoint $F^{\mathbb{T}}$ and the monad induced by $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ is $\mathbb{T}$.*

Proof. We define $F^{\mathbb{T}}A = (TA, \mu_A)$, this is a $\mathbb{T}$ algebra by the two diagrams in the definition of $\mathbb{T}$. If $f : A \to B$, we take $F^{\mathbb{T}}f = Tf : (TA, \mu_A) \to (TB, \mu_B)$, this is a homomorphism by the naturality of μ. To verify $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$, we construct the unit and counit of the adjunction. $G^{\mathbb{T}}F^{\mathbb{T}} = T$, so we take $\eta : 1 \to T$ as the unit. We define $\epsilon_{(A,\alpha)} = \alpha$, the associativity condition says that this a homomorphism $F^{\mathbb{T}}G^{\mathbb{T}}(A, \alpha) \to (A, \alpha)$ and naturality follows from the conditions on the morphisms of $\mathbb{T}$ algebras. The identity

$$\begin{tikzcd} G^{\mathbb{T}}(A,\alpha) \arrow[r, "\eta_A"] \arrow[dr, "1"'] & G^{\mathbb{T}}F^{\mathbb{T}}G^{\mathbb{T}}(A,\alpha) \arrow[d, "G^{\mathbb{T}}\alpha"] \\ & G^{\mathbb{T}}(A,\alpha) \end{tikzcd}$$

is the unit condition on α for $\mathbb{T}$ algebras. The identity

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "1"'] & FGFA \arrow[d, "\eta_{FA}"] \\ & FA \end{tikzcd}$$

is the diagram

$$\begin{tikzcd} TA \arrow[r, "T\eta_A"] \arrow[dr, "1"'] & TTA \arrow[d, "\mu_A"] \\ & TA \end{tikzcd}$$

which is included in the definition of monad. $\square$

Note that if $\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}$ is an adjunction inducing $\mathbb{T}$, we could replace $\mathcal{D}$ by the full subcategory $\mathcal{D}'$ on objects of the form FA. So in trying to construct $\mathcal{D}$ we may assume F is surjective. Also morphisms $FA \to FB$ in $\mathcal{D}$ correspond to morphisms $A \to GFB = TB$ in $\mathcal{C}$.

5.4 Definition. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The Kleisli category, $C_{\mathbb{T}}$ is defined by $\operatorname{ob}\mathcal{C}_{\mathbb{T}} = \operatorname{ob}\mathcal{C}$. Morphisms $A \boldsymbol{\to} B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \to TB$ in $\mathcal{C}$. The composite of $A \xrightarrow{\boldsymbol{f}} B \xrightarrow{\boldsymbol{g}} C$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC,$$

with the identity morphism $A \boldsymbol{\to} A$ as $A \to TA$. To verify associativity, suppose that we are given

$$A \xrightarrow{\boldsymbol{f}} B \xrightarrow{\boldsymbol{g}} C \xrightarrow{\boldsymbol{h}} D,$$

then the composite

$$\begin{aligned}
&\boldsymbol{h(gf)}\\
=&A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD\\
=&A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD\\
=&A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD\\
=&A \xrightarrow{f} TB \xrightarrow{T(\boldsymbol{hg})} TTD \xrightarrow{\mu_D} TD\\
=&\boldsymbol{(hg)f}
\end{aligned}$$

For the unit law,

$$\begin{aligned}
&A \xrightarrow{\boldsymbol{\eta_A}} A \xrightarrow{\boldsymbol{f}} B\\
=&A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB\\
=&A \xrightarrow{f} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} TB\\
=&A \xrightarrow{f} TB\\
=&A \xrightarrow{\boldsymbol{f}} B
\end{aligned}$$

The other identity is easier.

5.5 Lemma. *There is an adjunction* $\mathcal{C} \underset{F_\mathbb{T}}{\overset{G_\mathbb{T}}{\leftrightarrows}} \mathcal{C}_\mathbb{T}$*,* $F_\mathbb{T} \dashv G_\mathbb{T}$ *inducing the monad* $\mathbb{T}$*.*

Proof. We define $F_\mathbb{T}A = A$ and $F_\mathbb{T}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. To verify that $F_\mathbb{T}$ is a functor, suppose that we are given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$. Then $\boldsymbol{(F_\mathbb{T}g)(F_\mathbb{T}f)}$ is the composite

$$A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC = A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC = F_\mathbb{T}(gf)$$

We define $G_\mathbb{T}A = TA$, and $G_\mathbb{T}(A \xrightarrow{\boldsymbol{f}} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. We need to verify that $G_\mathbb{T}$ is a functor: First, the identities functions $G_\mathbb{T}(\eta_A) = \mu_A T\eta_A = 1_{TA}$, then

$$\begin{aligned}
&G_\mathbb{T}(A \xrightarrow{\boldsymbol{f}} B \xrightarrow{\boldsymbol{g}} C)\\
=&TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC\\
=&TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC\\
=&TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC\\
=&\boldsymbol{(G_\mathbb{T}g)(G_\mathbb{T}f)}
\end{aligned}$$

Note that $G_\mathbb{T}F_\mathbb{T}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TB$, so we take the unit of the adjunction to be $\eta : 1_\mathcal{C} \to T = G_\mathbb{T}F_\mathbb{T}$. $F_\mathbb{T}G_\mathbb{T}A = TA$, so we let the counit $TA \xrightarrow{\boldsymbol{\epsilon_A}} A$ be $TA \xrightarrow{1_{TA}} TA$. To verify the naturality of the counit consider:

$$\begin{tikzcd}
F_\mathbb{T}G_\mathbb{T}A \arrow[r, dotted, "F_\mathbb{T}G_\mathbb{T}f"] \arrow[d, dotted, "\epsilon_A"'] & F_\mathbb{T}G_\mathbb{T}B \arrow[d, dotted, "\epsilon_B"] \\
A \arrow[r, dotted, "f"'] & B
\end{tikzcd}$$

The top composition is

$$\begin{aligned}
=&TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{T1_{TB}} TTB\mu_B TB\\
=&TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TTB\mu_B TB\\
=&TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB
\end{aligned}$$

The bottom composition is

$$TA \xrightarrow{1_{TA}} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$$

so the square commutes. Finally we need to verify the triangular identities:

$$\begin{tikzcd}
F_\mathbb{T}A \arrow[r, dotted, "F_\mathbb{T}\eta_A"] \arrow[dr, dotted, "1_{F_\mathbb{T}A}"'] & F_\mathbb{T}G_\mathbb{T}F_\mathbb{T}A \arrow[d, dotted, "\epsilon_{F_\mathbb{T}A}"] \\
 & F_\mathbb{T}A
\end{tikzcd}$$

is

$$A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = A \xrightarrow{\eta_A} TA = \mathbf{1}_{\boldsymbol{F_{\mathbb{T}}A}}$$

The other triangular identity is

$$\begin{tikzcd} G_{\mathbb{T}}A \arrow[r, "\eta_{G_{\mathbb{T}}A}"] \arrow[dr, "1_{G_{\mathbb{T}}A}"'] & G_{\mathbb{T}}F_{\mathbb{T}}G_{\mathbb{T}}A \arrow[d, "G_{\mathbb{T}}\epsilon_A"] \\ & G_{\mathbb{T}}A \end{tikzcd}$$

but this is just

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "1_{TA}"'] & TTA \arrow[d, "\mu_A"] \\ & TA \end{tikzcd}$$

which commutes by definition of a monad. $\square$

Given a monad $\mathbb{T}$ on $\mathcal{C}$, let $\mathbf{Adj}(\mathbb{T})$ denote the category whose objects are adjunctions $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ with $F \dashv G$ inducing $\mathbb{T}$ and whose morphisms are functors

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "K"] \arrow[dr, shift left, "G"] & & \mathcal{D}' \arrow[dl, shift left, "G'"] \\ & \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F'"] & \end{tikzcd}$$

such that $KF = F'$ and $G'K = G$.

5.6 Theorem. *The Kleisli adjunction* $\mathcal{C} \underset{F_{\mathbb{T}}}{\overset{G_{\mathbb{T}}}{\leftrightarrows}} \mathcal{C}_{\mathbb{T}}$ *is initial in* $\boldsymbol{Adj}(\mathbb{T})$ *and the Eilenburg-Moore adjunction is terminal.*

Proof. Let $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ be an arbitrary object of $\mathbf{Adj}(\mathbb{T})$. Let ϵ be the counit of $F \dashv G$. We define $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$. Note that

$$\begin{tikzcd} GB \arrow[r, "\eta_G B"] \arrow[dr] & TGB \arrow[d, "G\epsilon_B"] \\ & GB \end{tikzcd}$$

commutes and

$$\begin{tikzcd} TTGB \arrow[r, "G\epsilon_{FGB} = \mu_G B"] \arrow[d, "GFG\epsilon_B"'] & TGB \arrow[d, "G\epsilon_B"] \\ TGB \arrow[r, "G\epsilon_B"'] & GB \end{tikzcd}$$

commute by naturality of ϵ. We let $K(B \xrightarrow{g} B') = Gg'$ (which is an algebra homomorphism since ϵ is natural). Clearly $G^{\mathbb{T}}K = G$ and $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ with $KF(A \xrightarrow{f} A') = GFf = Tf = F^{\mathbb{T}}f$.

If $K' : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ satisfies $G^{\mathbb{T}}K' = G$ and $K'F = F^{\mathbb{T}}$, then necessarily $K'B = (GB, \beta_B)$ and $K'g = Gg$ for some $\beta : GFG \to G$. Moreover, $\beta_{FA} = \mu_A = G\epsilon_{FA}$ for all $A \in \operatorname{ob}\mathcal{C}$. For any B we have $\epsilon_B : FGB \to B$ in $\mathcal{D}$ yielding

$$\begin{tikzcd} GFGFGB \arrow[r, "GFG\epsilon_B"] \arrow[d, "\mu_{GB}=G\epsilon_{FGB}"'] & GFGB \arrow[d, "\beta_B"] \\ GFGB \arrow[r, "G\epsilon_B"'] & GB \end{tikzcd}$$

but this diagram would commute if β_B was replaced by $G\epsilon_B$, but $GFG\epsilon_B$ is split epic, so $\beta_B = G\epsilon_B$.

Now for the Kleisli category. Define $L : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ by $LA = FA$, $L(A \xrightarrow{\boldsymbol{f}} A') = FA \xrightarrow{Ff} FGFA' \xrightarrow{\epsilon_{FA'}} FA'$. Check that L is a functor. $LF_{\mathbb{T}}(A \xrightarrow{f} A') = FA \xrightarrow{Ff} FA' \xrightarrow{F\eta_{A'}} FGFA' \xrightarrow{\epsilon_{FA'}} FA' = Ff$, $GLA = GFA = TA = G_{\mathbb{T}}A$, $GL(A \xrightarrow{\boldsymbol{f}} A') = GFA \xrightarrow{GFf} GFGFA' \xrightarrow{G\epsilon_{FA'}} GFA' = TA \xrightarrow{Tf} TTA' \xrightarrow{\mu'_A} TA' = G_{\mathbb{T}}f$ (proof of uniqueness left out). $\square$

5.7 Theorem. *Let $\mathbb{T}$ be a monad on $\mathcal{C}$. Then*

(i) $G^{\mathbb{T}} : \mathcal{C}^{TT} \to \mathcal{C}$ create limits of all shapes with exist in $\mathcal{C}$.

(ii) $G^{\mathbb{T}}$ creates colimits of shape J iff T preserves them.

Proof. (i) Suppose $D : J \to \mathcal{C}^{\mathbb{T}}$ is a diagram and $G^{\mathbb{T}}D$ has a limit $(L \xrightarrow{\lambda_i} G^{\mathbb{T}}D(j) : j \operatorname{ob} J)$ in $\mathcal{C}$. Write $D(j)$ as $(G^{\mathbb{T}}D(j), \delta_j)$. Then the $T\lambda_j$ forms a cone over $TG^{\mathbb{T}}D$ and the δ_j form a natural transformation $TG^{\mathbb{T}}D \to G^{\mathbb{T}}D$ since the $D(f : i \to j)$ are $\mathbb{T}$ algebra morphisms. Thus the composites $TL \xrightarrow{T\lambda_j} TG^{\mathbb{T}}D(j) \xrightarrow{\delta_j} G^{\mathbb{T}}D(j)$ form a cone over $G^{\mathbb{T}}D$. Hence we get a unique $\theta : TL \to L$ such that $\lambda_j\theta = \delta_j(T\lambda_j)$ for each j. We claim that (L, θ) is a $\mathbb{T}$-algebra. To verify, we need to show that two diagrams commute. We need to show that $\theta \circ (T\theta) = \theta \circ \mu_L$. First compose each with λ_j, then we get

$$\lambda_j \circ \theta \circ (T\theta) = \delta_j \circ (T\lambda_j) \circ (T\theta) = \delta_j \circ T(\delta_j \circ (T\lambda_j)) = \delta_j \circ (T\delta_j) \circ (TT\lambda_j)$$
$$\lambda_j \circ \theta \circ \mu_L = \delta_j \circ (T\lambda_j) \circ \mu_L = \delta_j \circ \mu_{G^{\mathbb{T}}D(j)} \circ (TT(\lambda_j))$$

But $D(j) = (G^{\mathbb{T}}, \delta_j)$ is an algebra, so $\delta_j \circ (T\delta_j) = \delta_j \circ \mu_{G^{\mathbb{T}}D(j)}$, thus the two compositions are the same cone over $G^{\mathbb{T}}D$, so their factorization through the limit is the same, that is $\theta \circ (T\theta) = \theta \circ \mu_L$. For the other diagram, we must show that $\theta \circ \eta_L = 1_L$, so we apply the same argument:

$$\lambda_j \circ \theta \circ \eta_L = \delta_j \circ (T\lambda_j) \circ \eta_L = \delta_j \circ \eta_{G^{\mathbb{T}}D(j)} \circ \lambda_j$$
$$1_L \circ \lambda_j$$

As above $D(j)$ is an algebra, so $\delta_j \circ \eta_{G^{\mathbb{T}}D(j)} = 1_L$, so these are factorizations of the same cone through the limit, hence $\theta \circ \eta_L = 1_L$.

It is clear that $((L, \theta) \xrightarrow{\lambda_i} D(i))$ is a cone, since we have $\lambda_j \theta = \delta_j(T\lambda_j)$ the morphism condition for $\mathbb{T}$-algebras. Finally we claim that (L, θ) is actually a limit for $D(j)$ in $\mathcal{C}^{\mathbb{T}}$. Given a cone $(M \xrightarrow{\mu_j} D(j) : j \in \operatorname{ob} J)$ in $\mathcal{C}^{\mathbb{T}}$, we get a unique factorization $\mu_j = \lambda_j \varphi$ viewing the cone over $\mathcal{C}$. All we need to show is that φ is a $\mathbb{T}$ algebra homomorphism, that is $\theta \circ (T\varphi) = \varphi \circ \beta$, we start by composing with λ_j:

$$\begin{aligned} \lambda_j \circ \theta \circ (T\varphi) = \delta_j \circ (T\lambda_j) \circ (T\varphi) &= \delta_j \circ (T\mu_j) \\ \lambda_j \circ \varphi \circ \beta &= \mu_j \circ \beta \end{aligned}$$

But μ_j is a $\mathbb{T}$ algebra homomorphism, so these two are equal. Thus they are a factorization of the same cone through a limit, and φ is a $\mathbb{T}$-algebra homomorphism.

(ii) The reverse implication is essentially a dual copy of the previous argument, but we use the fact that if L is the summit of a colimit cone, then so are TL and TTL. For the forward implication, if $G^{\mathbb{T}}$ creates colimits of shape J, then $T = G^{\mathbb{T}}F^{\mathbb{T}}$ preserves them since $F^{\mathbb{T}}$ preserves all colimits that exist. $\square$

5.8 Definition. We sat an adjunction $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ $F \dashv G$ is monadic if the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ where $\mathbb{T}$ is the monad induced by $F \dashv G$ is part of an equivalence. We also say $G : \mathcal{D} \to \mathcal{C}$ is monadic if it has a left adjoint and the adjunction is monadic.

Given an adjunction $F \dashv G$ for any object B of $\mathcal{D}$, we have a diagram

$$FGFGB \underset{FG\epsilon_B}{\overset{\epsilon_{FGB}}{\rightrightarrows}} FGB \xrightarrow{\epsilon_B} B$$

called the standard free presentation of B. The monadicity theorems all use the idea that $\mathcal{C}^{\mathbb{T}}$ is characterized in $\mathbf{Adj}(\mathbb{T})$ by the fact that this diagram is a coequilizer for any B.

5.9 Definition. We say a parallel pair $A \underset{f}{\overset{g}{\rightrightarrows}} B$ is reflexive if there exists $B \xrightarrow{r} A$ such that $fr = gr = 1_B$. By a reflexive coequalizer we mean a coequalizer of a reflective pair. We say a diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f{,}g"] \arrow[r, shift left=1] & B \arrow[l, shift left=1, "t"] \arrow[r, shift left=1, "h"] & C \arrow[l, shift left=1, "s"] \end{tikzcd}$$

is a split coequalizer diagram if it satisfies $hf = hg$, $hs = 1_C$, $gt = 1_b$ and $ft = sh$. Note that if these hold, then h is indeed a coequalizer of f and g: if $B \xrightarrow{k} D$, satisfies $kf = kg$, then $k = kgt = kft = ksh$, so k factors through h and the factorization is unique since h is split epic. Note that split equilizers are then preserved by any functor.

Given $G : \mathcal{D} \to \mathcal{C}$ we say that $A \underset{f}{\overset{g}{\rightrightarrows}} B$ is G-split if $GA \underset{Gf}{\overset{Gg}{\rightrightarrows}} GB$ is part of a split coequalizer diagram. Note that the standard free presentation $FGFGB \underset{FG\epsilon_B}{\overset{\epsilon_{FGB}}{\rightrightarrows}} FGB$ is reflexive with common splitting $F\eta_{GB}$ and is also G-split since

$$\begin{tikzcd} GFGFGB \arrow[r, shift left=2, "{Gf,Gg}"] \arrow[r, shift left=0.7] & GFB \arrow[l, shift left=0.7, "\eta_{GFGB}"] \arrow[r, shift left, "G\epsilon_B"] & GB \arrow[l, shift left, "\eta_{GB}"] \end{tikzcd}$$

is a split coequalizer diagram.

5.10 Theorem. *(Perfect Monadicity Theorem) Let $G : \mathcal{C} \to \mathcal{C}$ be a functor. Then G is monadic if:*

(i) G has a left adjoint

(ii) G creates coequalizers of G-split pairs

5.11 Theorem. *(Crude Monadicity Theorem) Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and suppose:*

(i) G has a left adjoint

(ii) G reflects isomorphisms

(iii) G preserves and $\mathcal{D}$ has coequalizers of reflective pairs

then G is monadic.

Proof. Proof of 5.10, forward direction follows from 5.7*ii*. Since $\mathbb{T}$ must preserve split coequalizers, so $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$ creates $G^{\mathbb{T}}$-split coequalizers (see proof of Beck's Theorem in Saunders-Maclean). But what does this tell us about G? Note that $G = G^{\mathbb{T}}K$, so if a pair $A \underset{f}{\overset{g}{\rightrightarrows}} B$ is a G-split coequalizer, then it is $G^{\mathbb{T}}K$-split coequalizer. Thus there is an coequalizer for the pair $G^{\mathbb{T}}KA \rightrightarrows G^{\mathbb{T}}KB \xrightarrow{e} C$, but $G^{\mathbb{T}}$ creates such colimits, so there exists a C', $e' : KB \to C'$ such that $KA \rightrightarrows KB \xrightarrow{e'} C'$ is a coequalizer. Now, K is full, faithful and essentially surjective, so there exists some C'' such that $KC'' \cong C'$ and an e'' the composition of e' and the isomorphism. Thus this is a colimit for the pair $A \underset{f}{\overset{g}{\rightrightarrows}} B$, since we can push any other cone down via K and pull the factorization back up since K is full.

Proof of 5.11 and 5.10 reverse direction. We have $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ with $\mathbb{T}$ the monad induced by the adjunction $F \dashv G$. Define $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$ by setting $L(A, \alpha)$ to be the coequilizer of $FGFA \underset{F\alpha}{\overset{\epsilon_{FA}}{\rightrightarrows}} FA$ (note that this is reflexive, since $F\eta_A$ is a common splitting and is G split since $GFGFA \underset{GF\alpha}{\overset{G\epsilon FA}{\rightrightarrows}} GFA \to A$ with

$\eta_{GFA} : GFA \to GFGFA$ and $\eta_A : A \to GFA$ as a split coequalizer). On morphisms L is defined such that

$$\begin{tikzcd} FGFA \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "FGFf"'] & FA \arrow[r] \arrow[d, "Ff"'] & L(A,\alpha) \arrow[d, "Lf"'] \\ FGFB \arrow[r, shift left] \arrow[r, shift right] & FB \arrow[r] & L(B,\beta) \end{tikzcd}$$

commutes. This is clearly functorial.

$$\begin{tikzcd} KFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "\mu_A"'] & KFA \arrow[r, "\alpha"] \arrow[dr] & (A,\alpha) \arrow[d] \\ & & KL(A,\alpha) \end{tikzcd}$$

is a $G^{\mathbb{T}}$-split coequalizer diagram. So we get a unique factorization $(A,\alpha) \to KL(A,\alpha)$ which is natural in A.

$KB = (GB, G\epsilon_B)$, so we have a coequalizer diagram

$$\begin{tikzcd} FGFGB \arrow[r, shift left, "FG\epsilon_B"] \arrow[r, shift right, "\epsilon_{FGB}"'] & FGB \arrow[r] \arrow[dr, "\epsilon_B"'] & LKB \arrow[d] \\ & & B \end{tikzcd}$$

hence we get a unique factorization $LKB \to B$ that is natural in B. The unit $(A,\alpha) \to KL(A,\alpha)$ maps to an isomorphism $A \to GL(A,\alpha)$ in $\mathcal{C}$ provided G preserves the coequalizer defining L. But $G^{\mathbb{T}}$ reflects isomorphisms, so it must be an isomorphism in $\mathcal{C}^{\mathbb{T}}$. Similarly $LKB \to B$ maps to an isomorphism in $\mathcal{C}$, so if G reflects isomorphisms, or if G creases the coequalizer of $FGFGB \rightrightarrows FGB$ then $LKB \to B$ must be an isomorphism. $\square$

5.12 Examples.

(a) For any category of algebras (in the universal algebra sense), e.g. **Rng**, **Gp**, etc... the forgetful functor to **Set** is monadic. The left adjoint exists (free groups, etc...) and the functor reflects isomorphisms, we know that if $A_1 \underset{f_1}{\overset{g_1}{\rightrightarrows}} B_1 \xrightarrow{h} C_1$ and $A_2 \underset{f_2}{\overset{g_2}{\rightrightarrows}} B_2 \xrightarrow{h_2} C_2$ are coequalizers in **Set** then $A_1 \times A_2 \underset{(f_1,f_2)}{\overset{(g_1,g_2)}{\rightrightarrows}} B_1 \times B_2 \xrightarrow{(h_1,h_2)} C_1 \times C_2$ is a coequalizer. Note that two elements of B_i are identified in C_i if we can link them by a chain where each adjoint pair is in the image of either $A_i \xrightarrow{(f_i,g_i)} B_i \times B_i$ or $A_i \xrightarrow{(g_i,f_i)} B_i \times B_i$. So if we have strings $b_{1,1}, ..., b_{1,2}$ and $b_{2,1}, ..., b_{2,2}$ we can link $(b_{1,1}, b_{2,1})$ to $(b_{1,2}, b_{2,2})$ since both pairs are reflective. Hence if $A \underset{f}{\overset{g}{\rightrightarrows}} B \xrightarrow{h} C$ is a reflective coequalizer in **Set**, so is $A^n \rightrightarrows B^n \to C^n$ for any finite n. A and B have have an n-ary operation and f, g are homomorphisms for it, we get a unique $C^n \to C$ making h a homomorphism. This shows that $U : \mathcal{A} \to \mathbf{Set}$ creates and reflects coequalizers.

(b) Any reflection is monadic. Direct proof in question 3 example sheet 3 but it ca be proved using 5.10: Suppose $\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ is reflection, identify $\mathcal{D}$ with a full subcategory of $\mathcal{C}$. If $A \underset{f}{\overset{g}{\rightrightarrows}} B$ is a G-split pair in $\mathcal{D}$, we have a split coequalizer diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "{f,g}"] \arrow[r, shift left=0.7] & B \arrow[l, shift left=0.7, "t"] \arrow[r, shift left, "h"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

in $\mathcal{C}$ since G is the inclusion of $\mathcal{D}$ in $\mathcal{C}$. We know that $sh : B \to B$ is in D since G is full and faithful. Note that $s : C \to B$ is an equalizer of $B \underset{sh}{\overset{1_B}{\rightrightarrows}} B$ since $shs = s1_C = s = 1_B s$, but we claim that if $\mathcal{D}$ is reflective in $\mathcal{C}$, then it is closed under limits in $\mathcal{C}$:

Suppose that we have a $D : J \to \mathcal{D}$, then GD is a diagram in $\mathcal{C}$. Suppose that it has a limit cone in $\mathcal{C}$, that is $(L \xrightarrow{\lambda_j} GD(j) : j \in \mathrm{ob}\, J)$. Suppose that we have any cone $(M \xrightarrow{\mu_j} D(j) : j \in \mathrm{ob}\, J)$ in $\mathcal{D}$, then applying G gives us a cone over GD in $\mathcal{C}$, so it factors through the limit L via $G\mu_j = \lambda_j \varphi$ in $\mathcal{C}$. Then by naturality of ϵ the following commutes:

$$\begin{tikzcd}
 & & & M \arrow[d, "\epsilon_M^{-1}"'] \arrow[ddddlll, "\mu_j"'] \arrow[ddddrrr, "\mu_i"] & & & \\
 & & & FGM \arrow[d, "F\varphi"'] \arrow[ddl, "FG\mu_j"'] \arrow[ddr, "FG\mu_i"] & & & \\
 & & & FL \arrow[dl, "F\lambda_j"] \arrow[dr, "F\lambda_i"'] & & & \\
 & & FGD(j) \arrow[rr, "FGDf"] \arrow[dll, "\epsilon_{D(j)}"] & & FGD(i) \arrow[drr, "\epsilon_{D(i)}"'] & & \\
D(j) \arrow[rrrrrr, "Df"] & & & & & & D(i)
\end{tikzcd}$$

in $\mathcal{D}$. Thus FL is a limit cone over $D(j)$ in $\mathcal{D}$.

(c) Consider the composite adjunction $\mathbf{Set} \underset{F}{\overset{U}{\leftrightarrows}} \mathbf{AbGp} \underset{L}{\overset{I}{\leftrightarrows}} \mathbf{tfAbGp}$. Each factor is monadic by 5.12a and 5.12b, but the composite is not, since free abelian groups are torsion free and so the monad on **Set** is induced by $(LF \dashv UI)$ is isomorphic to that induced by $(F \dashv U)$. In general, given an adjunction

$\mathcal{C} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{D}$ where $\mathcal{D}$ has reflexive coequalizers, we can form the 'monadic tower' where $\mathbb{T}$ is the monad induced by $(F \dashv G)$ and L is the left adjoint to the comparision functor K.

$$\begin{tikzcd}
\mathcal{D} \arrow[rrr, shift left] \arrow[drr, shift left, "K"] \arrow[ddr, shift left, "G"] & & & (\mathcal{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[lll, shift left] \arrow[dl, shift left] \\
& & \mathcal{C}^{\mathbb{T}} \arrow[ull, shift left, "L"] \arrow[ur, shift left] \arrow[dl, shift left, "G^{\mathbb{T}}"] & \\
& \mathcal{C} \arrow[uul, shift left, "F"] \arrow[ur, shift left, "F^{\mathbb{T}}"] & &
\end{tikzcd}$$

$\mathbb{S}$ is the monad induced by $L \dashv K$ and so on. We say $F \dashv G$ has monadic length n if this produced an equivalence after n steps. So $\mathbf{Set} \leftrightarrows \mathbf{tfAbGp}$ has monadic length ∞.

(d) Consider the adjunction $\mathbf{Set} \underset{D}{\overset{U}{\leftrightarrows}} \mathbf{Top}$ where D is the discretization functor. The monad induced by this adjunction is $(1_{\mathbf{Set}}, 1, 1)$ so its category of algebras is isomorphic to $\mathbf{Set}$, hence the adjunction has monadic length ∞.

(e) Consider the composite adjunction $\mathbf{Set} \underset{D}{\overset{U}{\leftrightarrows}} \mathbf{Top} \underset{\beta}{\overset{I}{\leftrightarrows}} \mathbf{kHaus}$. This is monadic, E Manes gave a direct proof, we will use 5.10. We need to show that UI creates coequalizers of UI split pairs. Suppose $X \underset{f}{\overset{g}{\rightrightarrows}} Y$ is a parallel pair in $\mathbf{kHaus}$, and

$$\begin{tikzcd}
X \arrow[r, shift left=2, "{f,g}"] \arrow[r, shift left=0.7] & Y \arrow[l, shift left=0.7, "t"] \arrow[r, shift left, "h"] & Z \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a split equalizer diagram in $\mathbf{Set}$. We need to show that there is a unique compact hausdorf topology on Z such that h is continuous and that it is then a coequalizer in $\mathbf{kHaus}$. We can think of Z as the quotient Y/R for a relation to be defined. So if we equip this with the quotient topology, we get a coequalizer in $\mathbf{Top}$. The quotient in $\mathbf{Top}$ is certainly compact since its the continuous image of a compact set. So it is the only topology making h continuous which could possibly be hausdorf.

We shall use this fact from topology: If Y is compact hausdorf and $R \subset Y \times Y$ is an equivalence relation then Y/R is hausdorf iff R is closed in $Y \times Y$. Claim: the equivalence relation R generated by $\{(f(x), g(x) : x \in X\}$ is the set $\{(g(x_1), g(x_2) : x_1, x_2 \in X, f(x_1) = f(x_2)\}$ It is clear that R is contained in this set. If $(y_1, y_2) \in R$, then $h(y_1) = h(y_2)$, so $ft(y_1) = sh(y_1) = sh(y_2) = ft(y_2)$ so $y_i = g(x_i)$ with $x_i = t(y_i)$ and $f(x_1) = f(x_2)$. The set $\{(x_1, x_2) | f(x_1) = f(x_2)\}$ is closed in $X \times X$, and hence compact, so its image under $g \times g$ is compact in $Y \times Y$ and hence closed.

6 Abelian Categories

6.1 Definition. Let $\mathcal{A}$ be a category equipped with a forgetful functor $U : \mathcal{A} \to$ **Set**. Say a locally small category $\mathcal{C}$ is enriched over $\mathcal{A}$ if we are given a factorization of $\mathcal{C}(-,-) : \mathcal{C}^{\mathbf{op}} \times \mathcal{C} \to \mathbf{Set}$ through U. If $\mathcal{A} = \mathbf{Set}_*$ we sat that $\mathcal{C}$ is a pointed category. If $\mathcal{A} = \mathbf{CMon}$, we say that $\mathcal{C}$ is semi-additive. If $\mathcal{A} = \mathbf{AbGp}$ we say $\mathcal{C}$ is abelian.

6.2 Lemma.

(i) If $\mathcal{C}$ is pointed, $I \in \operatorname{ob}\mathcal{C}$, the following are equivalent:

  (a) I is initial

  (b) I is terminal

  (c) $1_I = 0 : I \to I$

(ii) If $\mathcal{C}$ is semiadditive, $A, B, C \in \operatorname{ob}\mathcal{C}$ the following are equivalent

  (a) There exists π_1, π_2 making C into a product $A \times B$

  (b) There exists ν_1, ν_2 making C into a coproduct $A + B$.

  (c) There exists morphisms $\pi_1 : C \to A$, $\pi_2 : C \to B$, $\nu_1 : A \to C$ and $\nu_2 : B \to C$ such that $\pi_1\nu_1 = \pi_2\nu_2 = 1_B$, $\pi_2\nu_1 = 0$, $\pi_1\nu_2 = 0$ and $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$.

Proof. See examples sheet 2 □

6.3 Lemma. *Suppose $\mathcal{C}$ is a locally small category with finite products and coproducts such that $0 \to 1$ is an isomorphism. Given $A + B$ and $C \times D$, and*

$$\begin{array}{ll} f : A \to C & h : B \to D \\ k : A \to D & g : B \to D \end{array}$$

we can define a morphism $A + B \to C \times D$ by factoring the cocone $C \times D$ with $(f, k) : A \to C \times D$ and $(h, g) : B \to C \times D$. Now, let $f = 1_A$, $g = 1_B$ and $h = k = 0$ where 0 is the unique morphism factoring through 0, then we have $\varphi : A + B \to A \times B$. Suppose φ is an isomorphism, then $\mathcal{C}$ has a unique semi-additive structure.

Proof. The 0 of the semi-additive structure has to be defined as in the lemma. We need $0f = f0 = 0$ for all f, but this is clear since if $f : A \to B$ and $0 : B \to C$, then $0 = 0_C 0_B$ where $0_B : B \to 0$, so we must have $0_B f = 0_A$ so $0f = 0_C 0_B f = 0_C 0_A = 0 : A \to C$ similarly for $f0$. Given $A \underset{f}{\overset{g}{\rightrightarrows}} B$, we define $f +_l g$ to be $A \xrightarrow{(f,g)} B \times B \to B + B \to B$ where the final function is the factorization of the cocone B under the coproduct. We define $f +_r g$ to be $A \xrightarrow{(1_A, 1_A)} A \times A \to A + A \to B$ with the final function the factorization of the cocone B with functions f and g through the colimit $A + A$. We claim that $f +_l 0 = f$ so we need to show that

$$\begin{tikzcd}
A \arrow[r, "{(f,0)}"] \arrow[dr] \arrow[ddrr, bend right, "f"'] & B \times B \arrow[rr] & & B + B \arrow[r] & B \\
& B \times 0 \arrow[u] \arrow[rr] \arrow[dr] & & B + 0 \arrow[u] \arrow[ur] & \\
& & B \arrow[ur] \arrow[uurr, bend right, "1_B"'] & &
\end{tikzcd}$$

commutes. Given for morphisms $f, g, h, k : A \to B$, let φ be the map they define from $A + A \to B \times B$. Then consider

$$\begin{aligned}
&A \xrightarrow{(1_A,1_A)} A \times A \to A + A \xrightarrow{\varphi} B \times B \to B + A \to B \\
=&A \xrightarrow{(1_A,1_A)} A \times A \to A + A \xrightarrow{(f +_l h, g + lk)} B \\
=&(f +_l h) +_r (g +_l k)
\end{aligned}$$

But evaluating another way, we get $(f +_l h) +_r (g +_l k) = (f +_r g) +_l (h +_r k)$ hence $+_r = +_l$ and they are associative and commutative.

For the uniqueness, reacall from 6.2 that if we have any semi-additive structure, the identity map $A \times A \to A \times A$ is equal to $\nu_1\pi_1 + \nu_2\pi_2$. So, given $A \underset{f}{\overset{g}{\rightrightarrows}} B$, the composite

$$\begin{aligned}
&A \xrightarrow{(1,1)} A \times A \to A + A \xrightarrow{(f,g)} B \\
=&A \xrightarrow{(1,1)} A \times A \xrightarrow{\nu_1\pi_1 + \nu_2\pi_2} A \times A \to A + A \xrightarrow{(f,g)} B \\
=&A \xrightarrow{\nu_1 + \nu_2} A + A \xrightarrow{(f,g)} B \\
=&A \xrightarrow{f+g} B
\end{aligned}$$

□

An object which is both initial and terminal is called a zero object . An object which is both a product $A \times B$ and a coproduct $A + B$ is called a biproduct and denoted $A \oplus B$. We will feel free to use the matrix of functions f, h, g, k to denote a morphism $A \oplus B \to C \oplus D$ for biproducts. Note that composition is matrix multiplication.

6.4 Corollary. *Let $\mathcal{C}$ and $\mathcal{D}$ be semi-additive categories with finite products. Then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite product iff it preserves addition (i.e. $F(0) = 0$ and $F(f + g) = Ff + Fg$).*

Proof. If F preserves $+$, then it preserves biproducts by 6.2. The converse follows from 6.3. □

6.5 Definition. Let $\mathcal{C}$ be a pointed category. By a kernel of a morphuism $A \xrightarrow{f} B$ we mean an equilizer of $A \underset{f}{\overset{0}{\rightrightarrows}} B$ (dually the cokernel). We say a morphism is normal if it occurs as a kernel. We say $A \xrightarrow{f} B$ is a pseudo-monomorphism if $fg = 0$ implies $g = 0$ (equivalently the kernel of f is $0 : 0 \to A$). If $\mathcal{C}$ is addative, then every regular monomorphism is normal, since the equalizer $A \underset{f}{\overset{g}{\rightrightarrows}} B$ has the same universal property as the kernel of $A \xrightarrow{f+g} B$. And every pseudo-monomorphism is monic, since $fg = fh$ iff $f(g-h) = 0$. In **Gp** every monomorphism is regular, but a monomorphism $H \to G$ is normal if H is a subgroup of G, but every epimorphism $f : G \to K$ is normal since if f is surjective, then $K \cong G/\ker f$. In **Set** every monomorphism is normal, since if $A \xrightarrow{f} B$ is injective, then it is the kernel of $B \to B/\sim$ where $b_1 \sim b_2$ iff $b_1 = b_2$ or $\{b_1, b_2\} \subset \operatorname{im} f$. But not every epimorphism in **Set** is normal.

6.6 Lemma. *Let $\mathcal{C}$ be a pointed category with cokernels, then $f : A \to B$ is a normal monomorphism in $\mathcal{C}$ iff $f = \ker \operatorname{coker} f$.*

Proof. The reverse implication is trivial. For the forward implication, suppose that $f = \ker(g : B \to B)$, Let $q = \operatorname{coker} f$, then $gf = 0 = g0$, so g coequalizes f and 0, hence it factors as hq. Now given $k : E \to B$ with $qk = 0$, we have $gk = hqk = 0$ so there is a unique factorization $k = fl$. Since $qf = 0$, this implies $f = \ker q$ since f is a now a equalizer of 0 and q.

$$\begin{tikzcd}
E \arrow[d, "l"'] \arrow[dr, "k"] & & \\
A \arrow[r, "f"'] & B \arrow[r, "f"] \arrow[dr, "q"'] & C \\
& & D \arrow[u, "h"']
\end{tikzcd}$$

$\square$

6.7 Lemma. *Suppose $\mathcal{C}$ is pointed with kernels and cokernels and every monomorphism in $\mathcal{C}$ is normal. Then every morpism of $\mathcal{C}$ factors as a pseudo epimorphism followed by a monomorphism and the factorization is unique up to isomorphism.*

Proof. Given $A \xrightarrow{f} B$, let $q : B \to C$ be the cokernel of f and let $k : D \to B$ be the kernel of q. We get a factorization $f = kg$, we claim that g is a pseudo-epimorphism. Suppose $h : D \to E$ satisfies $hg = 0$, let $l = \ker h$, then kl is monic so $kl = \ker m$ for some m. We can factor g as ln, so $f = kg = kln$, so $mf = 0$, so $m = pq$ for some p. Now $qk = 0$ since $k = \ker q$ so $mk = 0$, so k factors through

kl. But k and l are monic, so this forces l to be an isomorphism and hence $h = 0$.

$$\begin{tikzcd}
 & & & G & \\
A \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[dd, "n"'] & & B \arrow[ur, "m"] \arrow[rr, "q"'] & & C \arrow[ul, "p"'] \\
 & D \arrow[ur, "k"'] \arrow[dr, "h"] & & & \\
F \arrow[ur, "l"] & & E & &
\end{tikzcd}$$

For uniqueness, suppose f factors as kg where g is pseudo-epimorphic, then $\operatorname{coker} f = \operatorname{coker} k$, so if k is also a monomorphism, then $k = \ker \operatorname{coker} k = \ker \operatorname{coker} f$ by 6.6. □

6.8 Definition. An abelian category is an additive category with finite limits and colimits (equally finite biproducts, kernels and cokernels) in which every monomorphism and every epimorphism is regular (equivalently normal).

AbGp, $\mathbf{Mod}_R$, $[\mathcal{C}, \mathcal{A}]$ where $\mathcal{A}$ is abelian. if $\mathcal{C}$ is additive and $\mathcal{A}$ is abelian, then the subcategory $\mathbf{Add}(\mathcal{C}, \mathcal{A}) \subset [\mathcal{C}, \mathcal{A}]$ of additive functors $\mathcal{C} \to \mathcal{A}$ is abelian. Note that $\mathbf{Mod}_R = \mathbf{Add}(R, \mathbf{AbGp})$ where we consider a ring R as an additive category with one object.

In a pointed category with kernels and cokernels, we write $\Im f$ for $\ker \operatorname{coker} f$ and $\operatorname{coim} f$ for $\operatorname{coker} \ker f$. In an abelian category, and f factors as $(\operatorname{im} f)g$ from the proof of 6.7 and as $h(\operatorname{coim} f)$ from the dual of the proof of 6.7, but both of these are factorings as an epimorphism followed by a monomorphism, so by 6.7 these must be the same. In general, we get a comparison map $\tilde{f}$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "\tilde{f}"'] & D
\end{tikzcd}$$

and in an abelian category $\tilde{f}$ is an isomorphism. Note that $\mathcal{A}$ is abelian iff $\mathcal{A}$ is additive with finite limits, colimits and every f factors as $(\operatorname{im} f)(\operatorname{coim} f)$. The forward implication is clear, for the reverse implication, if f is a monomorphism, $\operatorname{coim} f = \operatorname{coker} \ker f = \operatorname{coker}(0 \to A) = 1_A$, so $f = \operatorname{im} f$ and is normal by 6.6, similarly for epimorphisms.

6.9 Lemma. *Suppose that we are given a pullback square in an abelian category* *with h epic. Then the square is also a pushout and g is epic.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

Proof. Consider $A \xrightarrow{(f,-g})B \oplus C \xrightarrow{(h,k)} D$, we have $hf - gk = 0$ so the composite is 0 and the fact that $(f, -g)$ as the universal property of the pullback implies that $(f, -g) = \ker(h, k)$. But h is epic, so (h, k) is epic thus $(h, k) = \operatorname{coker}\ker(h, k) = \operatorname{coker}(f, -g)$ by 6.6. Thus the original square is a pushout.

Now consider $q = \operatorname{coker} g$ with $q : C \to E$. Then q and $0 : B \to E$ form a cone under

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

so they factor uniquely through D by $r : D \to E$. Then $rh = 0$, but h is epic, so $r = 0$ thus $q = rk = 0$, so g is epic. $\square$

6.10 Definition. We say a sequence of morphisms

$$... \to A \xrightarrow{g} B \xrightarrow{f} C \to ...$$

is exact at B if $\operatorname{im} g = \ker f$ or equivalently $\operatorname{coker} g = \operatorname{coim} f$. We can say that $f : A \to B$ is monic iff $0 \to A \xrightarrow{f} B$ is exact. Similarly f is epic iff $A \xrightarrow{f} B \to 0$ is exact.

A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is exact iff it preserves exactness of sequences. We say F is left exact if it preserves exactness of the form $0 \to A \to B \to C$ (i.e. it also preserves kernels) and F is right exact if it preserves the exactness of $A \to B \to C \to 0$ (i.e. it also preserves cokernels). By considering the exact sequence $0 \to A \xrightarrow{(1,0)} A \oplus B \xrightarrow{(0,1)} B \to 0$ and $0 \to B \xrightarrow{(0,1)} A \oplus B \xrightarrow{(1,0)} A \to 0$. We see that any left exact functor F must preserve biproducts: since it is left exact, it preserves the exactness except at the last nontrivial term. But note that it must carry 0 to 0, so $FB \to 0$ must have kernel 1_{FB}, on the other hand, the last non trivial function is split epic, which F must also preserve, hence its image is epic and thus has image 1_{FB}. Similar arguments hold for right exact functors. Hence F is left exact iff F preserves all finite limits. Also F is exact iff F preserves kernels and cokernels iff F preserves all finite limits and colimits.

6.11 Lemma. *(Five Lemma) Suppose that we are given a diagram*

$$
\begin{tikzcd}
A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "u_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "u_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "v_1"'] & B_2 \arrow[r, "v_2"'] & B_3 \arrow[r, "v_3"'] & B_4 \arrow[r, "v_4"'] & B_5
\end{tikzcd}
$$

with exact rows and f_1 is an epimorphism, f_2 and f_4 are isomorphisms and f_5 is a monomorphism, then f_3 is an isomorphism.

Proof.

$$
\begin{tikzcd}
M \arrow[rr, "d"] \arrow[ddr, bend right=60, "m"'] & & L \arrow[r, "e"] \arrow[d, "l"'] & K \arrow[d, "k"] & & \\
N \arrow[u, "c"] \arrow[r, "n"] & A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "u_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "u_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
& B_1 \arrow[r, "v_1"'] & B_2 \arrow[r, "v_2"'] & B_3 \arrow[r, "v_3"'] & B_4 \arrow[r, "v_4"'] & B_5
\end{tikzcd}
$$

First we show that f_3 is monic. Let $k : K \to A_3$ be the kernel of f_3. Now, $f_4 u_3 k = v_3 f_3 k = 0$ and f_4 is monic, so $u_3 k = 0$. Thus k facts through $\ker u_3 = \operatorname{im} u_2$ that is $k = (\operatorname{im} u_2)z$. Hence form the pullback of k and u_2 as e and l, note that it is also the pullback of $\operatorname{coim} u_2$ and z since the following diagram commutes

$$
\begin{tikzcd}
A_3 & K \arrow[l, "k"'] \arrow[d, "z"] \\
A_2 \arrow[u, "u_2"] \arrow[r, "\operatorname{coim} u_2"'] & I \arrow[ul, "\operatorname{im} u_2"]
\end{tikzcd}
$$

so e is epic. Now $v_2 f_2 l = f_3 u_3 l = f_3 k e = 0$, so $f_2 l$ factors through $\ker v_2 = \operatorname{im} v_1$m form the pullback of $f_2 l$ and v_1 to get d and m with d epic. Finally, form the pullback of m and f_1, giving n and c and since f_1 is epic, so is c. So $f_2 l d c = v_1 m c = v_1 f_1 n = f_2 u_1 n$, f_2 is epic so $ldc = u_1 n$, now $kedc = u_2 ldc = u_2 u_1 n = 0$, but edc is epic, so $k = 0$, hence f is monic, dually f is epic, and these are both regular, so f is an isomorphism. $\square$

6.12 Lemma. *Snake Lemma (see handout)*

6.13 Definition. By a complex in an abelian category $\mathcal{A}$ we mean a sequence of objects

$$... \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to ...$$

of objects and morphisms such that $d_n d_{n+1} = 0$ for all n. Nogte that this is just an additive functor $\mathcal{Z} \to \mathcal{A}$ where $\mathcal{Z} = \mathbb{Z}$, $\mathcal{Z}(n,n) = \mathcal{Z}(n, n+1) = \mathbb{Z}$ and $\mathcal{Z}(n,m) = 0$ for $m \neq n, n+1$. Here the complexes in $\mathcal{A}$ are the objects of an abelian category $c\mathcal{A} = \mathbf{Add}(\mathcal{Z}, \mathcal{A})$. Given a complex $C.$ we define $Z_n \to C_n$ to be the kernel of d_n, $B_n \to C_N$ to be the image of d_{n+1} and $Z_n \to H_n$ the cokernel of $B_n \to Z_n$. Equivalently we could form $C_n \to A_n$ as the cokernel of d_{n+1} and then $Z_n \to H_n \to A_n$ is the image factorization of $Z_n \to C_n \to A_n$. Each of $C. \to Z_n$, $C. \to B_n$, $C. \to A_n$ and $C. \to H_n$ defines an additive functor $c\mathcal{A} \to \mathcal{A}$. Note that $H_n = 0$ iff $C.$ is exact at C_n.

6.14 Theorem. *(Mayer-Vietoris) Suppose that we are given an exact sequence $0 \to C'. \to C. \to C''. \to 0$ in $c\mathcal{A}$, then there is an exact sequence*

$$... \to H'_n \to H_n \to H''_n \to H'_{n-1} \to H_{n-1} \to H''_{n-1} \to ...$$

of homology objects in $\mathcal{A}$.

Proof. First consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & Z'_n \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & Z''_n \arrow[d] & \\
 & C'_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & C''_{n-1} \arrow[d] & \\
 & A'_{n-1} \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & Z''_{n-1} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By 6.12 the rows $Z'_n \to Z_n \to Z''_n$ and $A'_{n-1} \to A_{n-1} \to A''_{n-1}$ are exact. Moreover, $Z'_n \to Z_n$ is monic, since $Z'_n \to Z_n \to C_n = Z'_n \to C'_n \to C_n$ is monic. Similarly, $A_{n-1} \to A''_{n-1}$ is epic. Now consider

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[rrr] \arrow[dr] & & & C_n \arrow[r] \arrow[dr] & \cdots \\
Z_{n+1} \arrow[ur] \arrow[rr] & & A_{n+1} \arrow[r] & Z_n \arrow[ur] \arrow[rr] & & A_n
\end{tikzcd}$$

is exact since $H_{n+1} \to A_{n+1} = \mathrm{im}(Z_{n+1} \to A_{n+1}) = \ker(A_{n+1} \to Z_n)$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H'_n \arrow[r] \arrow[d] & H_n \arrow[r] \arrow[d] & H''_n \arrow[d] \arrow[dddll, rounded corners] & \\
 & A'_n \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z'_{n-1} \arrow[r] \arrow[d] & Z_{n-1} \arrow[r] \arrow[d] & Z''_{n-1} \arrow[d] & \\
 & H'_{n-1} \arrow[r] \arrow[d] & H_{n-1} \arrow[r] \arrow[d] & H''_{n-1} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

By 6.12 we get a morphism $H''_n \to H'_{n-1}$ making the sequence

$$H'_n \to H_n \to H''_n \to H'_{n-1} \to H_{n-1} \to H''_{n-1}$$

exact. $\square$

7 Monoidal Categories and Closed Categories

7.1 Examples. We frequently encounter instances of a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $I \in \mathrm{ob}\,\mathcal{C}$ which makes $\mathcal{C}$ into a monoid with $\otimes = \times$ and $I = 1$.

(a) Any category with finite product taking $\otimes = \times$ and $I = 1$. We know that $A \times (B \times C) \cong (A \times B) \times C$ and $1 \times A \cong A \cong A \times 1$ since they are limits of the same diagram. Similarly, any category with finite coproducts with $\otimes = +$ and $I = 0$.

(b) In **AbGp** we have the usual tensor product $\otimes$ with $I = \mathbb{Z}$. In $\mathbf{Mod}_R$ we can do the same when R is commutative.

(c) For any $\mathcal{C}$ we have a monoidal structure on $[\mathcal{C}, \mathcal{C}]$ where $\otimes$ is composition of functors with $I = 1_{\mathcal{C}}$.

(d) Consider the category $\mathbf{\Delta}$ with $\mathrm{ob}\,\mathbf{\Delta} = \mathbb{N}$, with maps $n \to m$ order preserving maps from $\{0, ..., n-1\}$ to $\{0, ..., m-1\}$ with the operation being addition. Although $n + m = m + n$, this is not a natural isomorphism.

7.2 Definition. By a monoidal structure on a category $\mathcal{C}$, we mean a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object I equipped with natural isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$ such that 'all diagrams consisting of α, η and ρ commute'. In particular, we ask that the following diagrams commute:

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[rr, "\alpha_{A,B,C\otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B\otimes C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
& (A \otimes (B \otimes C)) \otimes D \arrow[ur, "\alpha_{A,B,C}\otimes 1_D"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha A{,}I{,}B"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}$$

Note that for $(\mathbf{AbGp}, \otimes, \mathbb{Z})$, the usual $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ sends a generator $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$. But we also have an isomorphism $\bar{\alpha}$ sending $a \otimes (b \otimes c)$ to $-(a \otimes b) \otimes c$, but $\bar{\alpha}$ does not satisfy the previous conditions.

7.3 Theorem. *(Coherence Theorem for Monoidal Categories) If the two diagrams above commute, then 'everything' does. More formally, we define the set of words in $\otimes$ and I as follows: we have a stock of variables $A, B, C, ...$ which are words, I is a word and if U and V are words, so is $(U \otimes V)$. If U, V, W are words, then $\alpha_{U,V,W} : (U \otimes (V \otimes W)) \to ((U \otimes V) \otimes W)$ is an instance of α (similarly for λ and ρ). If θ is an istance of α, λ, ρ, so are $1_A \otimes \theta$ and $\theta \otimes 1_A$. The body of a word is the sequence of variables. The theorem then says: given two words W and W' with the same body, there exists a unique morphism from $W \to W'$ which is obtained by composing instances and there inverses.*

Proof. Note that a word involving n variables defines a functor $\mathcal{C}^n \to \mathcal{C}$ and each α, λ or ρ defines a natural isomorphism between two such functors. We define a reduction step to be an instance of α, λ or ρ (as opposed to their inverses). Define the height of a word w, $h(w)$, to be $a(w) + i(w)$ where $i(w)$ is the number of occurrences of I and $a(w)$ is the number of instances of $\otimes$ occurring before a (.

Note that if $\theta : w \to w'$ is an instance of α, then $i(w) = i(w')$ and $a(w) > a(w')$. If θ is an instance of λ or ρ, then $i(w) > i(w')$ and $a(w) \geq a(w')$. Thus any sequence of reduction steps must terminate at a reduced word, from which no further reductions are possible. Clearly words of height 0, that is those of the form $((...(A_1 \otimes A_2) \otimes A_3...) \otimes A_n)$, and the word I of height 1 are reduced. These are the only reduced words: since if $i(w) > 0$ and $w \neq I$, then w has a subword $I \otimes u$ or $u \otimes I$, to which we can apply either λ or ρ. If $a(w) > 0$, then there is a substring $\otimes($ and hence a subword $(u \otimes (v \otimes z))$ to which we can apply α. For any w, any reduction path from w must lead to the reduced word with the same body, since we don not add or remove variables.

To show that any two reduction paths are equal as natural transformations, it suffices to prove that any pair of reductions $w' \xleftarrow{\varphi} w \xrightarrow{\theta} w''$ can be embedded in a commutative polygon, if so we can fill out the diagram of the two reduction paths with smaller polygons that are commutative, hence making the exterior polygon commutative. We consider the following cases:

1. φ and θ operate on disjoint subwords. That is $w = (u \otimes v)$ and $\theta = (\theta' \otimes 1_v)$ and $\varphi = (1_u \otimes \varphi')$, then the following diagram commutes by the functoriality of $\otimes$

$$\begin{tikzcd} & (u \otimes v) \arrow[dl, "\theta' \otimes 1"'] \arrow[dr, "1 \otimes \varphi'"] & \\ (u' \otimes v) \arrow[dr, "1 \otimes \varphi'"'] & & (u \otimes v') \arrow[dl, "\theta' \otimes 1"] \\ & (u' \otimes v') & \end{tikzcd}$$

2. φ operates within one argument of θ. I.e. $\theta = \alpha_{u,v,z}$ and $\varphi = 1 \otimes (\varphi' \otimes 1)$ where $\varphi' : v \to v'$. Then we have

$$\begin{tikzcd} & (u \otimes (v \otimes z)) \arrow[dl, "\alpha"'] \arrow[dr, "1 \otimes (\varphi' \otimes 1)"] & \\ ((u \otimes v) \otimes z) \arrow[dr, "(1 \otimes \varphi') \otimes 1"'] & & (u \otimes (v' \otimes z)) \arrow[dl, "\alpha"] \\ & ((u \otimes v') \otimes z) & \end{tikzcd}$$

3. φ and θ interfere with each other. If θ and φ are both instances of α, w must contain $(u \otimes (v \otimes (x \otimes y)))$ and θ, φ are $\alpha_{u,v,x \otimes y}$ and $1 \otimes \alpha_{v,x,y}$ and we use the pentagonal diagram to get our commutative polygon. If θ is an α and φ is a λ, then either w contains $u \otimes (I \otimes v)$ and $\theta = \alpha_{u,I,v}$ and $\varphi = 1 \otimes \lambda_v$ and we use the triangular diagram to get out commutative polygon. Or, w contains a subword $I \otimes (u \times v)$, $\theta = \alpha_{I,u,v}$ and $\varphi = \lambda_{u \otimes v}$. For this we need to know that the following diagram commutes:

$$\begin{tikzcd} I \otimes (u \otimes v) \arrow[rr, "\alpha"] \arrow[dr, "\lambda_{u \oplus v}"'] & & (I \otimes u) \otimes v \arrow[dl, "\lambda_u \otimes 1"] \\ & u \otimes v & \end{tikzcd}$$

If θ is α and φ is ρ then w contains $u \otimes (v \otimes I)$, $\theta = \alpha_{u,v,I}$ and $\varphi = 1 \otimes \rho_v$. So we need to know that

$$\begin{tikzcd} u \otimes (v \otimes I) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\rho_v"'] & & (u \otimes v) \otimes I \arrow[dl, "\rho_{u\otimes v}"] \\ & u \otimes v & \end{tikzcd}$$

commutes. If θ is a λ and φ is a ρ then w contains $I \otimes I$, and θ an instance of λ_I and φ is an instance of ρ_I, so we need to know that $\lambda_I = \rho_I$.

Finally, given two words w and w' and a morphism $\varphi : w \to w'$, which is a composite of reductions and their inverses we can form

$$\begin{tikzcd} w \arrow[r] \arrow[drrr, "\theta_w"'] & w_1 \arrow[drr, "\theta_{w_1}"] & w_2 \arrow[l] \arrow[dr, "\theta_{w_2}"] & w_3 \arrow[l] \arrow[r] \arrow[d, "\theta_{w_3}"] & w_4 \arrow[r] \arrow[dl, "\theta_{w_4}"] & \cdots \arrow[r] & w' \arrow[dlll, "\theta_{w'}"] \\ & & & w_0 & & & \end{tikzcd}$$

where θ_w is the reduction from w to the reduced word w_0, this commutes by the uniqueness of reductions, so $\varphi = (\theta_{w'})^{-1}(\theta_w)$ and is unique. $\square$

7.4 Definition. Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. By a symmetry for $\otimes$, we mean a natural transformation $\gamma_{A,B} : A \otimes B \to B \otimes A$ satisfying

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "1_A\otimes\gamma_{B,C}"'] & (A \otimes B) \otimes C \arrow[d, "\gamma_{A\otimes B,C}"] \\ A \otimes (C \otimes B) \arrow[d, "\alpha_{A,C,B}"'] & C \otimes (A \otimes B) \arrow[d, "\alpha_{C,A,B}"] \\ (A \otimes C) \otimes B \arrow[r, "\gamma_{A,C}\otimes 1"'] & (C \otimes A) \otimes B \end{tikzcd}$$

and

$$\begin{tikzcd} I \otimes A \arrow[rr, "\gamma_{I,A}"] \arrow[dr, "\lambda_A"'] & & A \otimes I \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes B \arrow[rr, "\gamma_{A,B}"] \arrow[dr, "1_{A\otimes B}"'] & & B \otimes A \arrow[dl, "\gamma_{B,A}"] \\ & A \otimes B & \end{tikzcd}$$

There is a coherence theorem for symmetric monoidal categories similar to 7.3 (but more delicate: note that $\gamma_{A,A} \neq 1_{A\otimes A}$ in general, although it is true for

$A = I$). A given monoidal category may have more then one symmtery. If $\mathcal{C} = \mathbf{AbGp}^{\mathbb{Z}}$ with $(A. \otimes B.)_n = \bigoplus_{p+q=n} A_p \otimes B_q$ and I_n defined as $\mathcal{Z}$ for $n = 0$ and 0 otherwise. We could define $\gamma_{A,B}$ to be the map $a \otimes b \mapsto b \otimes a$ or we could take $a \otimes b \mapsto (-1)^{pq} b \otimes a$ where $a \in A_p$ and $b \in B_q$. Both of these satisfy the digrams in 7.4

7.5 Definition. Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. By a (lax) monoidal structure on F we mean a natural transformation $\theta_{A,B} : FA \otimes FB \to F(A \otimes B)$ and a morphism $\iota : I \to FI$ such that the following diagrams commute:

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "1\otimes\theta_{B,C}"] \arrow[d, "\alpha_{FA,FB,FC}"'] & FA \otimes F(B \otimes C) \arrow[r, "\theta_{A,B\otimes C}"] & F(A \otimes (B \otimes C)) \arrow[d, "F\alpha_{A,B,C}"] \\
(FA \otimes FB) \otimes FC \arrow[r, "\theta_{A\otimes B}\otimes 1"'] & F(A \otimes B) \otimes C \arrow[r, "\theta_{A\otimes B,C}"'] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes FA \arrow[r, "\iota\otimes 1"] \arrow[d, "\lambda_{FA}"'] & FI \otimes FA \arrow[d, "\theta_{I\otimes A}"] \\
FA \arrow[r, "F\lambda_A"'] & F(I \otimes A)
\end{tikzcd}$$

and similarly for ρ. If the monoidal structures are symmetric, we say (θ, ι) is a symmetric monoidal structure if

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\theta_{A,B}"] \arrow[d, "\gamma_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\gamma_{A,B}"] \\
FB \otimes FA \arrow[r, "\gamma_{A,B}"'] & F(B \otimes A)
\end{tikzcd}$$

commutes. We say the monoidal structure is strong if θ and ι are isomorphisms.

Given monoidal functors (F, θ, ι) and (G, γ, κ), we say a natural transformation $\beta : F \to G$ is monoidal is

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\theta_{A,B}"] \arrow[d, "\beta_A\otimes\beta_B"'] & F(A \otimes B) \arrow[d, "\beta_{A\otimes B}"] \\
GA \otimes GB \arrow[r, "\gamma_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

commutes.

7.6 Examples.

(a) Let R be a commutative ring. The forgetful functor $(\mathbf{Mod}_R, \otimes_R, R) \to (\mathbf{AbGp}, \otimes, \mathbb{Z})$ is lax monoidal. If A and B are R-modules, we have a quotient map $A \otimes B \to A \otimes_R B$ and $\iota : Z \to R$ by $n \mapsto n1_R$.

(b) The functor $(\mathbf{AbGp}\otimes, \mathbb{Z}) \to (\mathbf{Set}, \times, 1)$ is lax monoidal. We take the universal map $A \times B \to A \otimes B$ for θ and $\iota : 1 \to Z$ picking out 1.

(c) The functor $\mathbf{AbGp} \to \mathbf{Mod}_R$ which sends A to $R \otimes A$ is strong monoidal: we have a canonical isomorphism $R \otimes \mathbb{Z} \cong R$ and $(R \otimes A) \otimes_R (R \otimes B) \cong R \otimes (A \otimes_R R) \otimes B \cong R \otimes (A \otimes B)$. In general given a monoidal adjunction (one whose unit and counit are monoidal natural transformations), between lax monoidal functors, the left adjoint is always strong. We get an inverse for $FA \otimes FB \to F(A \otimes B)$ from the composite

$$F(A\otimes B) \xrightarrow{F(\eta_A \otimes \eta_B)} F(GFA \otimes GFB) \to FG(FA \otimes FB) \xrightarrow{\eta_{FA\otimes FB}} FA \otimes FB$$

(d) If $(\mathcal{C}, \times, 1)$ and $(\mathcal{D}, \times, 1)$ are cartesian monoidal categories, then $F : \mathcal{C} \to \mathcal{D}$ is strong monoidal iff F preserves finite products.

$$\begin{tikzcd}
FA \times FB \arrow[r] \arrow[d] & F(A \times B) \arrow[d] \\
FA \times F1 \arrow[r] & F(A \times 1) \arrow[d] \\
FA \times 1 \arrow[u] \arrow[r] & FA
\end{tikzcd}$$

shows that θ commutes with projections.

(e) any functor betwen cocartesian categories has a unique lax monoidal structure and this structure is strong iff F preserves coproducts.

7.7 Definition. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. By a monoid in $\mathcal{C}$ we mean an object A equipped with morphisms $m : A \otimes A \to A$ and $e : I \to A$ such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[r, "1\otimes m"] \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[dd, "m"] \\
(A \otimes A) \otimes A \arrow[d, "m\otimes 1"'] & \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes A \arrow[r, "e\otimes 1"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "m"] & A \otimes A \arrow[l, "1\otimes e"'] \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

If $\otimes$ is symmetric, we say (A, m, e) is a commutative monoid if

$$\begin{tikzcd} A \otimes A \arrow[rr, "\gamma_{A,A}"] \arrow[dr, "m"'] & & A \times A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

also commutes.

7.8 Examples.

(a) In $(\mathbf{Set}, \times, 1)$ monoids are just monoids in the usual sense. Similarly we can consider any category with finite products e.g. **Top**. A monoid in **Cat** is a strict monoidal category (one for which α, λ, ρ are identities).

(b) In a cocartesian monoidal category $(\mathcal{C}, +, 0)$ every object has a unique (commutative) monoidal structure given by the unique morphism $0 \to A$ and the codiagonal map $A + A \to A$,

(c) In $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ (commutative) monoids are (commutative) rings.

(d) In $[\mathcal{C}, \mathcal{C}]$ monoids are monads.

(e) In $\mathbf{\Delta}$, the object 1 has a monoid structure given by the unique maps $0 \to 1$ and $1 + 1 = 2 \to 1$, This is the 'universal monoid', given any monoidal category $(\mathcal{C}, \otimes, I)$, the category of strong monoidal functors $\mathbf{\Delta} \to \mathcal{C}$ is equivalent to the category of monoids in $\mathcal{C}$ by the functor sending $F : \mathbf{\Delta} \to \mathcal{C}$ to $F(1)$ (note that given a monoid (A, m, e) in $\mathcal{B}$ and a (lax) monoidal functor $F : \mathcal{B} \to \mathcal{C}$, FA has a monoidal structure given by $FA \otimes FA \xrightarrow{\theta} F(A \otimes A) \xrightarrow{Fm} FA$ and $I \xrightarrow{\iota} FI \xrightarrow{Fe} FA$). Given a monoid (A, l, e) in $\mathcal{C}$, the morphisms $\overbrace{(...(A \otimes A) \otimes A...)}^{n} \to \overbrace{(...(A \otimes A) \otimes A...)}^{m}$ by composing instance of l and e correspond to the morphisms in $\mathbf{\Delta}(n, m)$

There is also a universal commutative monoid living in the category $\mathbf{Set}_f$ of finite sets and functions between them (with the cocartesian monoidal structure): it is the terminal object 1. Given a commutative monoid (A, m, e) in a arbitrary system, monoidal category $(\mathcal{C}, \otimes, I)$, the assignment $n \mapsto \overbrace{(...(A \otimes A) \otimes ...) \otimes A}^{n}$ can be made into a strong monoidal function $\mathbf{Set}_f \to \mathcal{C}$.

7.9 Definition. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. We say the monoidal structure is left closed, if, for each $A \in \operatorname{ob}\mathcal{C}$, $A \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint. Similarly $\otimes$ is right closed if $- \otimes A$ has a right adjoint. If both hold, we say $\otimes$ is biclosed. For a symmetric monoidal structure $\otimes$, we say $\otimes$ is closed if it is left (equivalently right) closed. We write $[A, -]$ for the right adjoint of $- \otimes A$. So we have natural bijections between $\mathcal{C}(A, [B, C])$ and $\mathcal{C}(A \otimes B, C)$, natural in A and C.

7.10 Examples.

(a) $(\mathbf{Set}, \times, 1)$ is closed. We say cartesian closed if $(\mathcal{C}, \times, 1)$ is closed. We know that functions $A \times B \to C$ correspond naturally to functions $A \to C^B$ where C^B is the set of functions $B \to C$, so we set $[B, C] = C^B$.

(b) **Cat** is cartesian closed. Here we take $[\mathcal{C}, \mathcal{D}]$ to be the category of all functors $\mathcal{C} \to \mathcal{D}$ and it is easy to see that functors $\mathcal{B} \to [\mathcal{C}, \mathcal{D}]$ correspond to functors $\mathcal{B} \times \mathcal{C} \to \mathcal{D}$.

(c) For any small category $\mathcal{C}$, $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed.

Proof. (1) Use the special adjoint functor theorem: $- \times F : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ preserves all small colimits, since limits and colimits are constructed pointwise by $4.7d$. We know $[\mathcal{C}, \mathbf{Set}]$ is cocomplete and locally small, has a separating set $\{\mathcal{C}(A, -) : A \in \operatorname{ob} \mathcal{C}\}$ and its well-copowered (since epimorphisms are pointwise surjective). $\square$

Proof. (2) Use the Yoneda lemma: whatever $[F, G]$ is, elements of $[F, G](A)$ must correspoind to natural transformations $\mathcal{C}(A, -) \times F \to G$. So we define $[F, G](A) = [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -) \times F, G)$. Given $f : A \to B$, we have $G(f, -) : \mathcal{C}(B, -) \to \mathcal{C}(A, -)$ and composition with $\mathcal{C}(f, -) \times 1_F$ yeilds a mapping $[F, G](A) \to [F, G](B)$. This makes $[F, G]$ into a functor. Verify that, for any H natural transformations $H \to [F, G]$ correspond bijectively to natural transformations $H \times F \to G$. $\square$

(d) $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ is closed: homomiorphisms $A \otimes B \to C$ correspond to bilinear maps $A \times B \to C$, which in turn correspond to homomorphisms $A \to \mathbf{AbGp}(B, C)$ where $\mathbf{AbGp}(B, C)$ is equipped with the pointwise abelian group structure, $(f + g)(b) = f(b) + g(b)$.