Relevant and Petite K-types

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• **NOTATION**
  - \( G \) is the real points of a linear connected reductive group.
  - \( g_0 := \text{Lie}(G) \), \( \theta \) Cartan involution, \( g := (g_0)_C \), \( g_0 = t_0 + s_0 \), \( K \) maximal compact subgroup.
  - A representation \((\pi, \mathcal{H})\) on a Hilbert space is called **unitary**, if \( \mathcal{H} \) admits a \( G \) invariant positive definite inner product.

• **PROBLEM**
  Classify all irreducible unitary representations of \( G \).
  By results of Harish-Chandra, it is enough to solve the

• **ALGEBRAIC PROBLEM**
  Classify all irreducible admissible unitary \((g, K)\) modules.
Irreducible admissible representations of $G$

- $P = MAN$ a parabolic subgroup of $G$, $a_0 := Lie(A)$, $a$ its complexification,
- $(\delta, V_\delta)$ an irreducible tempered unitary representation of $M$,
- $\nu \in a^*$, with real part in the open positive Weyl chamber,
- $X_P(\delta \otimes \nu)$ the corresponding Harish-Chandra induced (normalized induction) standard module,
- $\overline{X}_P(\delta \otimes \nu)$: the unique irreducible quotient.
Langlands, early 1970s:

- Every irreducible admissible representation of $G$ is infinitesimally equivalent to a **Langlands quotient** $X_P(\delta \otimes \nu)$.

- Two Langlands quotients $X_P(\delta \otimes \nu)$ and $X_{P'}(\delta' \otimes \nu')$ are infinitesimally equivalent if and only if there exists an element $\omega$ of $K$ such that
  \[ \omega P \omega^{-1} = P', \quad \omega \cdot \delta \cong \delta', \quad \omega \cdot \nu = \nu'. \]

- $X(\delta, \nu)$ is the image of an intertwining operator
  \[ A(\mathcal{P}, P, \delta, \nu) : X_P(\delta, \nu) \longrightarrow X_{\mathcal{P}}(\delta, \nu). \]
Knapp and Zuckerman, 1976:

$\mathcal{X}_P(\delta \otimes \nu)$ admits a **non-degenerate invariant Hermitian form** if and only if there exists $\omega \in K$ satisfying

\[
\omega P \omega^{-1} = \bar{P} \quad \omega \cdot \delta \simeq \delta \quad \omega \cdot \nu = -\bar{\nu}
\]

(because the Hermitian dual of $\mathcal{X}_P(\delta \otimes \nu)$ is $\mathcal{X}_P(\delta \otimes -\bar{\nu})$).

Any non-degenerate invariant Hermitian form on $\mathcal{X}_P(\delta \otimes \nu)$ is a real multiple of the form induced by the Hermitian operator

\[
B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} : P : \delta : \nu)
\]

from $X_P(\delta \otimes \nu)$ to $X_P(\delta \otimes -\bar{\nu})$.

$\mathcal{X}_P(\delta, \nu)$ is unitary if and only if $B$ is positive semidefinite.
The signature of $\mathcal{B}$

- For every $K$-type $(\mu, E_\mu)$, we have a Hermitian operator
  \[ R_\mu(\omega, \nu): \text{Hom}_K(E_\mu, X_P(\delta \otimes \nu)) \to \text{Hom}_K(E_\mu, X_P(\delta \otimes -\bar{\nu})). \]

- By Frobenius reciprocity:
  \[ R_\mu(\omega, \nu): \text{Hom}_{M \cap K}(E_\mu |_{M \cap K}, V^\delta) \to \text{Hom}_{M \cap K}(E_\mu |_{M \cap K}, V^\delta). \]

If $P$ is the minimal parabolic subgroup, and $\delta = \text{Triv}$, then

\[ R_\mu(\omega, \nu): (E_\mu^*)^M \longrightarrow (E_\mu^*)^M. \]
• $G$ is split, in particular $SL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$, $SO(n, n)$, $F_4$, $E_6$, $E_7$, $E_8$.

• $P = MAN$ is a minimal parabolic subgroup of $G$.

• $\delta$ is the trivial representation of $M$.

• $\nu$ is a real character of $A$.

In this case we can regard $\omega$ as an element of $W := N_K(a_0)/M$.

The operator $R_\mu(\omega, \nu)$ decomposes into a product of factors according to the decomposition of $\omega$ into a product of simple reflections (as in Gindikin-Karpelevic). These factors are induced from the corresponding intertwining operators on $SL(2, \mathbb{R})$. 
For each \( \alpha \in \Delta(n_0, a_0) \), choose a map \( \psi_\alpha : sl(2, \mathbb{R}) \rightarrow g_0 \) which commutes with \( \theta \), and satisfies
\[
\psi_\alpha \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = E_\alpha, \quad \psi_\alpha \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = E_{-\alpha},
\]
where \( E_{\pm \alpha} \) are the root vectors, and \( \theta(E_\alpha) = -E_{-\alpha} \). Then \( \psi_\alpha \) determines a map
\[
\Psi_\alpha : SL(2, \mathbb{R}) \rightarrow G
\]
with image \( G_\alpha \), a connected group with Lie algebra isomorphic to
Denote by 
\[ \sigma_\alpha := \Psi_\alpha \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad m_\alpha := \sigma_\alpha^2, \]
and let \( Z_\alpha := E_\alpha - E_{-\alpha} \in \mathfrak{t}_0. \)

**Definition.** A K-type is called **petite**, if \( \mu(iZ_\alpha) = 0, \pm 1, \pm 2, \pm 3. \)

The operators \( R_\mu(\omega, \nu) \) have a simpler form for such K-types. The factors corresponding to the simple root reflections are

\[
R_\mu(s_\beta, \nu) = \begin{cases} 
\frac{\nu}{\nu \cdot \beta} + 1 & \text{on the } (+1)\text{-eigenspace of } \mu(\sigma_\beta) \\
\frac{1}{\nu \cdot \beta} & \text{on the } (-1)\text{-eigenspace of } \mu(\sigma_\beta) 
\end{cases}
\]

The operator \( R_\mu(s_\beta, \nu) \) acts on \( (E_\mu^*)^M \), and depends only on the \( W \)-module structure of this space.
The formula for $R_{\mu}(s_{\beta}, \nu)$ coincides with the formula for the similar operator for a split p-adic group. To be more precise, results of Barbasch-Moy reduce the problem of the determination of the Iwahori spherical dual of split p-adic group to the problem of determining the unitary dual of finite dimensional representations of the corresponding affine graded Hecke algebra. In this case, for each representation $\tau \in \hat{W}$, there is an operator $R_\tau(\omega, \nu)$ with the same formula as the one for the real case. A spherical representation $\overline{X}(\nu)$ is unitary if and only if $R_\tau$ is positive definite for all $\tau$. 
Work of Barbasch for the classical groups, Ciubotaru for $F_4$, and Barbasch-Ciubotaru for $E_6$, $E_7$, and $E_8$, determine a set of $W$-representations, called relevant with the property that a spherical module $X(\nu)$ is unitary, if and only if $R_\tau(\omega, \nu)$ is positive semidefinite for $\tau$ in the relevant set.

**PROBLEM** Find a set of petite K-types so that the $(E_\mu^*)^M$ realize all the relevant $W$-representations.

If we can solve this problem, then we get powerful necessary conditions for unitarity in the real case. Conjecturally the spherical unitary dual for a split reductive group should be independent of whether the field is real or p-adic. This is true for the classical groups, but a conjecture for the exceptional groups.
For type $A_{n-1}$, $W = S_n$, and the relevant representations are

$$(n-k,k).$$

For types $B_n$, and $C_n$, the Weyl group $W$ consists of permutations and sign changes of the coordinates of $\mathbb{R}^n$, and the relevant $W$-types are

$$(n-k,k) \times (0), \quad (n-k) \times (k).$$

Similarly for $D_n$. 

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Exceptional Groups

The relevant $W$ representations are

$F_4$ \quad 1_1, 2_3, 8_1, 4_2, 9_1,$

$E_6$ \quad 1_6, 6_6, 20_p, 30_p, 15_q,$

$E_7$ \quad 1_a, 7_a, 27_a, 56'_a, 21'_b, 35_b, 105_b,$

$E_8$ \quad 1_x, 8_x, 35_x, 50_x, 84_x, 112_x, 400_x, 300_x, 210_x.$

The notation is as in Kondo’s and Frame’s work.
Let \((\mu_a, V_a)\) and \((\mu_b, V_b)\) be representations of \(K\). Then \(\text{Hom}_M[V_a, V_b]\) is endowed with a representation of \(N_K(M)\) via

\[n \cdot f(v) := \mu_b(n)f(\mu_a(n^{-1})v).\]

Under this action, \(M \subset N_K(M)\) acts trivially, so we get a representation of \(W\). Because

\[\text{Hom}_M[V_a, V_b] \cong \text{Hom}_M[V_a \otimes V_b^*, \text{Triv}],\]

this generalizes the action of \(W\) on \((E^*_\mu)^M\) from before.
Fine K-types

A K-type is called fine (Bernstein-Gelfand, Vogan), if
\[ \mu(iZ_\alpha) = 0, \pm 1. \]
These are the lowest K-types of principal series. A fine K-type has
the property that its restriction to \( M \) is multiplicity free, and is a
single \( N_K(a_0) \)-orbit of representations of \( M \).
In the case of a linear group, \( M \) is abelian, so \( \hat{M} \) is formed of
characters.
Fix a representative \( \delta \) for each \( W \)-orbit, and a fine K-type \( \mu_\delta \).
Then
\[ \mu_\delta \otimes \mu_\delta^* \text{ is formed of petite K-types only.} \]
We will use the previous formula to determine the Weyl group
representation on \( \mu_\delta \otimes \mu_\delta^* \).
Stabilizer of $\delta$

- $\vee \Delta^\delta := \{ \bar{\alpha} \mid \delta(m_{\alpha}) = 1 \}$ is a root system.
- The Weyl group generated by the roots in $\vee \Delta^\delta$ is called $W^{\circ}_\delta$, and is a normal subgroup of the stabilizer $W_\delta$ of $\delta$.
- The quotient $R_\delta := W_\delta/W^{\circ}_\delta$ is a product of $\mathbb{Z}_2$’s.
- $\hat{R}_\delta$ acts simply transitively on the fine $K$-types containing $\delta$.
- Inflate $\tau \in \hat{R}_\delta$ to $W_\delta$. Having fixed a $\mu_\delta$, there is a 1-1 correspondence

\[ \{ \tau \in \hat{W}_\delta \mid \tau|_{W^{\circ}_\delta} = \text{triv} \} \longleftrightarrow \{ \mu_{\delta, \tau} \}, \quad \text{triv} \longleftrightarrow \mu_\delta. \]

**Theorem.** As a $W$-module,

\[ \text{Hom}_M[\mu_{\delta, 1}, \mu_{\delta, \tau}] \cong \text{Ind}_{W^{\circ}_\delta}^W[\tau]. \]
Example 1

$G$ type $E_8$, $K = Spin(16)$. This is really the double cover of the rational points of the linear group. Let $\omega_i$ be the fundamental weights of $K$. In this case, $W_\delta = W_0^\delta$. The fine K-types are

<table>
<thead>
<tr>
<th>K-type</th>
<th>M-type</th>
<th>$W_\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0)$</td>
<td>$\delta_1$, trivial representation,</td>
<td>$W(E_8)$</td>
</tr>
<tr>
<td>$(\omega_1)$</td>
<td>$\delta_{16}$, dimension 16,</td>
<td>$W(E_8)$</td>
</tr>
<tr>
<td>$(\omega_2)$</td>
<td>$\delta_{120}$, 120 characters,</td>
<td>$W(E_7A_1)$</td>
</tr>
<tr>
<td>$(2\omega_1)$</td>
<td>$\delta_{135}$, 135 characters,</td>
<td>$W(D_8)$</td>
</tr>
</tbody>
</table>
In all cases, $W_\delta = W_\delta$.

\[ \text{Hom}_M[\mu_{\delta_{120}}, \mu_{\delta_{120}}] \cong \text{Ind}_{W(E_{\delta_{120}})}^{W(E_{\delta_{120}})}[\text{triv}] = 1_x + 35_x + 84_x, \]
\[ \text{Hom}_M[\mu_{\delta_{135}}, \mu_{\delta_{135}}] \cong \text{Ind}_{W(E_{\delta_{135}})}^{W(E_{\delta_{135}})}[\text{triv}] = 1_x + 84_x + 50_x. \]

It is straightforward that the reflection representation $8_2$ corresponds to the representation of $K$ on $s_0$:

\[ \omega_8 = 8_2 \delta_1 + \delta_{120}. \quad (1) \]

Quite a few relevant Weyl group representations do not occur in these two formulas.

The next tables give the Weyl representations on $(E_\mu)^M$ for $\mu$ petite.
## Petite K-types for E8

<table>
<thead>
<tr>
<th>K-type</th>
<th>W-type on $(E_8^\ast)^M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>$1_z$</td>
</tr>
<tr>
<td>$\omega_8$</td>
<td>$8_z$</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>$35_z$</td>
</tr>
<tr>
<td>$2\omega_2$</td>
<td>$84_z$</td>
</tr>
<tr>
<td>$\omega_2 + \omega_8$</td>
<td>$112_z$</td>
</tr>
<tr>
<td>$4\omega_1$</td>
<td>$50_z$</td>
</tr>
<tr>
<td>$3\omega_1 + \omega_7$</td>
<td>$400_z$</td>
</tr>
<tr>
<td>$2\omega_3$</td>
<td>$300_z$</td>
</tr>
<tr>
<td>K-type</td>
<td>W-type on $(E^s)^M$</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------</td>
</tr>
<tr>
<td>$\omega_3 + \omega_7$</td>
<td>160$_z$,</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>28$_x$,</td>
</tr>
<tr>
<td>$\omega_1 + \omega_5$</td>
<td>210$_x$,</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2 + \omega_7$</td>
<td>560$_z$,</td>
</tr>
<tr>
<td>$\omega_2 + \omega_4$</td>
<td>567$_x$,</td>
</tr>
<tr>
<td>$2\omega_1 + \omega_4$</td>
<td>700$_x$,</td>
</tr>
<tr>
<td>$3\omega_1 + \omega_3$</td>
<td>1050$_x$,</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2 + \omega_3$</td>
<td>1344$_x$,</td>
</tr>
<tr>
<td>$3\omega_2$</td>
<td>525$_x$,</td>
</tr>
<tr>
<td>$2\omega_1 + 2\omega_2$</td>
<td>972$_x$,</td>
</tr>
<tr>
<td>$4\omega_1 + \omega_2$</td>
<td>700$_{xx}$,</td>
</tr>
<tr>
<td>$6\omega_1$</td>
<td>168$_y$.</td>
</tr>
</tbody>
</table>
Only $\omega_1$ is genuine for $K = Spin(16)$, the others factor to a quotient group. In particular, genuine representations of $Spin(16)$ restrict to multiples of $\delta_{16}$. All representations are self dual. We compute

\[ \omega_2 \otimes \omega_2 = (2\omega_2) + (\omega_1 + \omega_3) + (2\omega_1) + (\omega_2) + (\omega_4) + (0), \]
\[ (2\omega_1) \otimes (2\omega_1) = (4\omega_1) + (2\omega_1 + \omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0). \]  

Furthermore, $\omega_3$ restricts to $35\delta_{16}$, and

\[ \omega_1 \otimes \omega_3 = (\omega_1 + \omega_3) + (\omega_2) + (\omega_4). \]

Thus the multiplicity of $\delta_1$ in $(\omega_1 + \omega_3) + (\omega_4)$ is 35. On the other hand, $\dim \omega_4 = 1820$, so the multiplicity of $\delta_1$ in $\omega_4$ is nonzero.
From (2) it follows that the multiplicity is exactly 35, and so
\[ \omega_4 \leftrightarrow 35_x. \]  
(4)

We also conclude that the multiplicity of \( \delta_1 \) in \( \omega_1 + \omega_3 \) is zero.

From the first equation in (2) we also conclude that \( (2\omega_2) \) contains \( \delta_1 \) 84 times, so
\[ (2\omega_2) \leftrightarrow 84_x. \]

Consider \( (\omega_1 + \omega_2) \) which restricts to \( 84\delta_{16} \). Then
\[ (\omega_1 + \omega_2) \otimes \omega_1 = (2\omega_1 + \omega_2) + (2\omega_1) + (\omega_1 + \omega_3) + (2\omega_2) + (\omega_2). \]  
(5)

Thus only \( 2\omega_2 \) contains \( \delta_1 \).

These arguments also imply
\[ Hom_M[\omega_1, \omega_3] \simeq 35_x. \]  
(6)

Combined with the second equation in \( (2) \) we get
\[ 4\omega_1 \leftrightarrow 50_x. \]  
(7)
We illustrate another aspect of the calculation. We know that $8z \otimes 50x = 400z$. Furthermore, assume that we have done some earlier calculations, and found that

\[
\text{Hom}_M[\omega_1, (3\omega_1)] \cong 50x,
\]
\[
\text{Hom}_M[\omega_1, \omega_7] \cong 8z,
\]
\[
\omega_2 + \omega_8 \leftrightarrow 112z.
\]

Then,

\[
(3\omega_1) \otimes (\omega_7) = (3\omega_1 + \omega_7) + (2\omega_1 + \omega_8),
\]
\[
(\omega_1 + \omega_8) \otimes \omega_1 = (2\omega_1 + \omega_8) + (\omega_1 + \omega_7) + (\omega_2 + \omega_8) + (\omega_8).
\] (8)

Since $\omega_1 + \omega_8 = 120\delta_{16}$, and $\omega_8$ contains eight copies of $\delta_1$, it follows that $\delta_1$ does not occur in $(2\omega_1 + \omega_8) + (\omega_1 + \omega_7)$. We conclude that

\[
(3\omega_1 + \omega_7) \longleftrightarrow 400z.
\] (9)
Petite K-types for $C_n$

$G = \text{Sp}(n, \mathbb{R})$, $K = U(n)$.

$\mu_+(k) := (1, \ldots, 1, 0, \ldots, 0)$, $\mu_-(k) := (0, \ldots, 0, -1, \ldots, -1)$

are fine K-types containing the same orbit of a character $\delta \in \hat{M}$.

The stabilizers are $W_0^\delta \cong W(D_k) \times W(C_{n-k})$, and

$W_\delta \cong W(C_k) \times W(C_{n-k})$. Then

$\text{Ind}_{W_\delta}^{W_0}[\text{triv}] = \sum (n - \ell, \ell) \times (0), \quad 0 \leq \ell \leq \min(k, n - k),$

$\text{Ind}_{W_\delta}^{W_0}[\tau] = (n - k) \times (k)$. 

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The tensor products are
\[ \mu_+(k) \otimes \mu_- (k) = \sum_{2a+b=2k} (1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1), \]
\[ \mu_+(k) \otimes \mu_- (k) = \sum_{2a+b=2k} (2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0). \]
These \( K \)-types are automatically petite, and in fact satisfy\[ \mu(iZ_a) = 0, \pm 1, \pm 2. \]
The precise correspondence is

- **K-type**
  - \((2, \ldots, 2, 0, \ldots, 0)\) \(\ell\xrightarrow{\ell} (n - \ell) \times (\ell)\)
  - \(\underbrace{1, \ldots, 1}_k, 0, \ldots, 0, -1, \ldots, -1\) \(\underbrace{k}_k\xrightarrow{k} (n - k, k) \times (0)\).

- **W-representation on \((E^*_\mu)^M\)**
The petite K-types with the property that $\mu(iZ_\alpha) = 0, \pm 1, \pm 2$, have some very nice properties. They are sufficient to determine unitarity in the classical cases, but not the exceptional ones.

Springer Correspondence

- $\mathfrak{g}$ complex semisimple Lie algebra, $\mathfrak{b} \subset \mathfrak{g}$ Borel subalgebra,
- $\mathcal{O} \subset \mathfrak{g}$ nilpotent orbit, $\{e, H, f\}$ Lie triple,
- $A(e)$ component group of the centralizer of $e$,
- $B_e := \{b \mid e \in b\}$, the incidence variety.

The Springer correspondence attaches to each $(\mathcal{O}, \psi \in \widehat{A(e)})$ a representation $\sigma(\mathcal{O}, \psi)$ of $W$ which is irreducible or zero. It is the representation of $W$ on $H^{top}(B_e)^\psi$, (maybe tensored with $sgn$ in
this case so that $\sigma((0), \text{triv}) = \text{triv} \in \hat{\W}$.

We have the following two assertions for $\mu$ petite, level 2.

- $\sigma \cong (E^*_\mu)^M$ if and only if the restriction of $\sigma$ to any rank two Levi does not contain $\text{sgn}$,
- $\sigma \cong (E^*_\mu)^M$ if and only if $\sigma = \sigma(O, \psi)$ where $O$ meets a Levi component with factors of type $A_1$ only.
Joint work with A. Pantano
Principal Series, Classical Groups

Type C

Consider

\[ \delta_k = (1, \ldots, 1, 0, \ldots, 0) \quad \leftrightarrow \quad \mu_k^+ = (1, \ldots, 1, 0, \ldots, 0) \]

\[ \mu_k^- = (0, \ldots, 0, -1, \ldots, -1) \]  (10)
Then the relevant K-types are

<table>
<thead>
<tr>
<th>K-type</th>
<th>$W(C_k \times C_{n-k})$-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$</td>
<td>$(\text{triv}) \otimes [(a, n - k - a) \times (0)]$</td>
</tr>
<tr>
<td>$(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$</td>
<td>$[(k - b) \times (b)] \otimes (\text{triv})$</td>
</tr>
<tr>
<td>$(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$</td>
<td>$(\text{triv}) \otimes [(n - k - b) \times (b)]$</td>
</tr>
<tr>
<td>$(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$</td>
<td>$[(b, k - b) \times (0)] \otimes (\text{triv})$.</td>
</tr>
</tbody>
</table>

(11)

We get another set of K-types by changing all the signs to minuses.

These K-types are petite because they are factors of the tensor
products

\[ \Lambda^r(\mathbb{C}^n) \otimes \Lambda^s(\mathbb{C}^n), \quad \text{or} \quad \Lambda^r(\mathbb{C}^n) \otimes \Lambda^s(\mathbb{C}^*)^n. \] (12)

**Type D**

These are the cases $SO(2n, 2n)$ and $SO(2n + 1, 2n + 1)$. For simplicity, just use $2n$, the other case is equivalent. Consider

\[ \delta_k = (1, \ldots, 1, 0, \ldots, 0) \quad \leftrightarrow \quad \mu^+_k = (1, \ldots, 1, 0, \ldots, 0) \otimes (0, \ldots, 0) \]

\[ \mu^-_k = (0, \ldots, 0) \otimes (0, \ldots, 0, -1, \ldots, -1) \]  

(13)
Then the relevant K-types are

\[(1,\ldots,1,0,\ldots,0) \otimes (1,\ldots,1,0,\ldots,0),\]
\[(1,\ldots,1,0,\ldots,0) \otimes (1,\ldots,1,0,\ldots,0),\]
\[(2,\ldots,2,1,\ldots,1,0,\ldots,0) \otimes (0,\ldots,0),\]
\[(2,\ldots,2,1,\ldots,1,0,\ldots,0) \otimes (0,\ldots,0).\]

We get another set of K-types with the same properties by interchanging the factors.
In the case of genuine principal series, only level $\leq 3/2$ K-types are petite. This has to do with the representation theory of the double cover $\widetilde{SL}(2,\mathbb{R})$. The matchings are
<table>
<thead>
<tr>
<th>K-type</th>
<th>M-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$1_x,$</td>
</tr>
<tr>
<td>$\omega_1 + \omega_2$</td>
<td>$84_x,$</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$35_x,$</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>$8_z,$</td>
</tr>
<tr>
<td>$3\omega_1$</td>
<td>$50_x.$</td>
</tr>
</tbody>
</table>

The missing ones are

<table>
<thead>
<tr>
<th>K-type</th>
<th>M-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1 + \omega_3$</td>
<td>$8_z + 112_z,$</td>
</tr>
<tr>
<td>$2\omega_1 + \omega_7$</td>
<td>$400_z + \ldots,$</td>
</tr>
<tr>
<td>$\omega_2 + \omega_3$</td>
<td>$300_x + \ldots,$</td>
</tr>
<tr>
<td>$\omega_1 + \omega_4$</td>
<td>$210_x + \ldots,$</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>$210_x + \ldots.$</td>
</tr>
</tbody>
</table>
\[ \delta_{135} \]

<table>
<thead>
<tr>
<th>K-type</th>
<th>W-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2\omega_1)</td>
<td>(8 \times 0)</td>
</tr>
<tr>
<td>(\omega_3)</td>
<td>(71 \times 0)</td>
</tr>
<tr>
<td>(4\omega_1)</td>
<td>(44 \times 0)</td>
</tr>
<tr>
<td>(\omega_1 + \omega_7)</td>
<td>(7 \times 1)</td>
</tr>
<tr>
<td>(2\omega_2)</td>
<td>(62 \times 0)</td>
</tr>
<tr>
<td>(2\omega_1 + \omega_4)</td>
<td>(6 \times 2)</td>
</tr>
<tr>
<td>(2\omega_1 + \omega_2)</td>
<td>(4 \times 4)</td>
</tr>
<tr>
<td>(\omega_2 + \omega_5)</td>
<td>(61 \times 1)</td>
</tr>
<tr>
<td>(\omega_1 + \omega_3)</td>
<td>(6 \times 2)</td>
</tr>
<tr>
<td>(2\omega_1 + \omega_8)</td>
<td>(5 \times 3 + 7 \times 1)</td>
</tr>
</tbody>
</table>

(17)
<table>
<thead>
<tr>
<th>K-type</th>
<th>W-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\omega_0)$</td>
<td>$6 \times 11$</td>
</tr>
<tr>
<td>$(2\omega_1 + \omega_4)$</td>
<td>$53 \times 0 + 4 \times 4_+ + 71 \times 0 + 51 \times 2 + 31 \times 4 + 42 \times 2$</td>
</tr>
<tr>
<td>$(\omega_1 + \omega_5)$</td>
<td>$4 \times 4_+ + 51 \times 2 + 6 \times 11 + 71 \times 0 + 6 \times 2$</td>
</tr>
</tbody>
</table>

**Theorem.** A parameter $(\delta_{135}, \nu)$ is unitary only if the corresponding spherical parameter $(\delta_1, \nu)$ is unitary for $D_8$. 
\[ \delta_{120} \]

<table>
<thead>
<tr>
<th>K-type</th>
<th>W-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_2 )</td>
<td>( 1_a \otimes 2 )</td>
</tr>
<tr>
<td>( 2\omega_2 )</td>
<td>( 21_a' \otimes 11 )</td>
</tr>
<tr>
<td>( \omega_1 + \omega_3 )</td>
<td>( 27_a \otimes 2 )</td>
</tr>
<tr>
<td>( \omega_8 )</td>
<td>( 1_a \otimes 11 )</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>( 7_a' \otimes 11 )</td>
</tr>
<tr>
<td>( \omega_1 + \omega_7 )</td>
<td>( 7_a' \otimes 2 )</td>
</tr>
<tr>
<td>( \omega_2 + \omega_8 )</td>
<td>( 27_a \otimes 11 + 21_a' \otimes 2 + \ldots )</td>
</tr>
<tr>
<td>( \omega_1 + \omega_5 )</td>
<td>( 56_a' \otimes 11 )</td>
</tr>
<tr>
<td>( 2\omega_1 + \omega_8 )</td>
<td>( 56_a' \otimes 2 )</td>
</tr>
<tr>
<td>( \omega_1 + \omega_2 + \omega_7 )</td>
<td>( 35_b \otimes 11 + \ldots )</td>
</tr>
</tbody>
</table>

(18)
Theorem. A parameter \((\delta_{120}, \nu)\) is unitary only if the corresponding spherical parameter \((\delta_1, \nu)\) is unitary for \(E_7 A_1\).
Let $\mu, \mu_1, \mu_2$ be genuine representations. The main point is that $\delta_{16}$ is the unique genuine representation of $M$, and it IS the irreducible $K$-module $\omega_1$.

- As a $W$-representation,
  $$\text{Hom}_M[\mu_1, \mu_2] \cong \text{Hom}_M[\mu_1, \omega_1] \otimes \text{Hom}_M[\mu_2, \omega_1].$$
  Decompose LHS as a $K$-module, RHS as a $W$-module.

- For $\delta = \delta_{120}$ or $\delta = \delta_{135}$, (irreducible representation of $M$)
  $$\text{Hom}_M[\delta, \omega_1 \otimes \mu] = \text{Res}_{W_\delta} \text{Hom}_M[\omega_1, \mu].$$
  Decompose $\omega_1 \otimes \mu$ as a $K$-module, the RHS as a $W_\delta$-module.