PELL'S EQUATION

\[ x^2 - Dy^2 = 1 \quad D > 0 \text{ not a square.} \]

RECALL: \[ a_0 = \lfloor \sqrt{D} \rfloor. \]

\[ \sqrt{D} + a_0 \text{ is purely periodic} \]

\[ \sqrt{D} + a_0 = \langle 2a_0, a_1 \ldots, a_m \rangle \]

\[ \sqrt{D} = \langle a_0, \ldots, a_m, 2a_0, a_1 \ldots a_m \rangle \]

\[ \sqrt{D} = \langle a_0, \ldots, a_m, \sqrt{D} + a_0 \rangle \]

\[ \sqrt{D} = \frac{p_m (\sqrt{D} + a_0) + p_{m-1}}{2_m (\sqrt{D} + a_0) + 2_{m-1}} = \frac{p_m \sqrt{D} + (p_m a_0 + p_{m-1})}{2_m \sqrt{D} + (2_m a_0 + 2_{m-1})} \]

WE GET: \[ \sqrt{D} = \frac{p_m \sqrt{D} + D \cdot 2_m}{2_m \sqrt{D} + p_m} \]

\[ \frac{p_m - D \cdot 2_m}{p_m} = (-1)^{m-1} \iff N(p_m + \sqrt{D} \cdot 2_m) = (-1)^{m-1} \]

If \( m \) even, we get a solution to \( x^2 - Dy^2 = -1 \)

Can do one of two things:

1. \[ (p_m \sqrt{D} + 2_m)^2 = (p_m^2 + D \cdot 2_m^2) + 2p_m \cdot 2_m \sqrt{D} \]

satisfies \( x^2 - Dy^2 = +1 \)

2. \[ \sqrt{D} = \langle a_0, \ldots, a_m, 2a_0, a_1 \ldots, a_m, \sqrt{D} + a_0 \rangle \]

Same argument as before yields
\[ \sqrt{D} = \frac{P_{2m+1}(\sqrt{D} + a_0) + P_{2m+1}}{2m+1(\sqrt{D} + a_0) + 2m+1} \]

\[ P_{2m+1} - 2m+1 \sqrt{D} = (-1)^m = 1. \]

**In fact,**

\[ \begin{align*}
P_{2m+1} &= P_{m} + D^{\frac{1}{2}} \cdot 2^m \\
2^{m+1} &= 2m \cdot 2^m \end{align*} \]

This comes from substituting \( \sqrt{D} = \frac{P_m \sqrt{D} + 1}{2m \sqrt{D} + 2^{m-1}} \)

\( \sqrt{D} = \langle a_0, \ldots, a_m, \sqrt{D} + a_0 \rangle \)

- Need to use

\( \begin{align*}
P_m a_0 + P_{m-1} &= D^{\frac{1}{2}} \cdot 2^m \\
2^m a_0 + 2^{m-1} &= P_m \end{align*} \)

and the similar relation for \( P_{2m+1}, 2^{m+1} \)

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**Thm:** Let \( x_0 + \sqrt{D}y_0 \) be the fundamental solution:

\( x_0 > 0, \ y_0 > 0 \) \& \( x + \sqrt{D}y \) any other,

\( x_0 < x \ \& \ y_0 < y \)

Then \( (x + \sqrt{D}y) = (x_0 + \sqrt{D}y_0)^k \) \( k \in \mathbb{N} \)

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**Recall:**

- Continuous fractions guarantee a solution
- If a sol'n exists, there is a unique fundamental one
- The fact that the first sol'n (in fact all) come from continued fractions is non-trivial
Proof: \[
\frac{x + \sqrt{D} y}{x_0 + \sqrt{D} y_0} = (x_0 x - y_0 D) + (x_0^2 - y_0 x) \sqrt{D} = u + v \sqrt{D}
\]

Want to show: 
1. \(0 < u < x\) \(\iff\) \(0 < x_0 x - y_0 D < x\)
2. \(0 < v < y\) \(\iff\) \(0 < x_0^2 - y_0 x < y\)

\(u^2 - D v^2 = 1\) is clear.

1. \(D \frac{y_0}{x_0} < \frac{x}{y}\)
2. \(\frac{1}{D} \frac{x}{y} < \frac{y_0}{x_0 - 1}\)
3. \(\frac{y_0}{x_0} < \frac{y}{x}\)
4. \(\frac{y}{x} < \frac{y_0}{x_0 - 1}\)

\(\frac{y_0}{x_0}\) slope of line \((0,0) \rightarrow (x_0, y_0)\)
\(\frac{y}{x}\) slope of line \((0,0) \rightarrow (x, y)\)
\(\frac{y_0}{x_0 - 1}\) slope of line \((1,0) \rightarrow (x_0, y_0)\)

3. comes from \(x^2 - Dy = 1\) concave down
1 \quad \frac{y}{x} < \frac{1}{\sqrt{D}} \iff \sqrt{D} < \frac{x}{y} \quad \frac{y_0}{x_0} < \frac{1}{\sqrt{D}}

\Rightarrow \quad D \frac{y_0}{x_0} < \frac{D}{\sqrt{D}} = \sqrt{D} < \frac{x}{y}

\textbf{4} \quad \text{comes from the picture}

\underline{1, 3, 4} \implies 2 \quad \Box