Approximations by Continuous Fractions

\[ <a_0, a_1, \ldots, a_m, \ldots> \quad a_m > 0 \quad m \geq 1 \]

\[ c_n = \frac{P_n}{Q_n} \quad \text{convergents}, \quad c_0 < c_1 < \cdots < c_k < c_0 \]

\[ \left| \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} \right| = \frac{1}{q_k q_{k-1}} \to 0 \quad \text{because} \quad \frac{q_k}{q_{k-1}} \geq 2^{k/2} \]

\[ \alpha \leftrightarrow <a_0, \ldots, a_{k-1}, \alpha_k> \]

\[ \alpha_k = a_k + \frac{1}{\alpha_{k+1}} \quad \text{and} \quad 1 < \alpha_{k+1} \]

**Basic Formula:** \[ \alpha - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k (\alpha_{k+1} q_k + q_{k-1})} \]

\[ \Rightarrow \left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k q_{k+1}} \]

Theorem: \[ \lim_{n \to \infty} c_n = \alpha \]

Corollary: If \[ <a_0, \ldots, a_m, \ldots> = <b_0, \ldots, b_m, \ldots> \]
then \[ a_i = b_i \]

**Proof:** \[ c_0 = a_0 = \frac{p_0}{q_0} < \lim < c_i = \frac{p_i}{q_i} = a_0 + \frac{1}{a_1} \]

It follows that \[ a_0 = \lfloor \lim \rfloor = b_0 \]

Next, note that \[ \lim = a_0 + \frac{1}{a_1} \]

\[ <a_1, \ldots> = \frac{1}{\lim - [a_0]} = \frac{1}{\lim - [b_0]} = <b_1, \ldots> \]

Use induction \[ \square \]

**Converse:** Suppose \( \frac{c}{d} \) is a fraction such that \[ |\alpha - \frac{c}{d}| < \frac{1}{2d^2} \]. Then \( \frac{c}{d} = \frac{p_k}{q_k} \) for some \( k \).
The proof is omitted; theorem 9.39.

Applications: \( \frac{2^2}{7} \) is the best approximation of \( \pi \) by a fraction with denominator \( \leq 7 \).

- Calendars

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Some Complements:

**Proposition:** \((p_k, q_k) = 1\).

**Proof:** \( p_k q_{k-1} - q_k p_{k-1} = (-1)^k \)

**Example:** \( \frac{34}{16} = 2 + \frac{2}{16} = 2 + \frac{1}{8} \)

\[ \frac{16}{2} = 8 \quad p_1 = 1 \quad p_0 = a_0 = 2 \quad p_1 = a_1 = 17 \]

\[ r_1 = 0 \quad q_0 = 1 \quad q_1 = 8 \]

\[ c_0 = 2 \quad c_2 \leq \frac{34}{16} \quad \leq c_1 = \frac{17}{8} \]

\[ x^2 - Dy^2 = \pm 1 \]. Suppose \( c + d\sqrt{D} \) is a solution.

\[ \frac{c}{d} - \sqrt{D} = \frac{c - d\sqrt{D}}{d} = \frac{c^2 - dD}{d(c + d\sqrt{D})} = \frac{\pm 1}{d^2(c + \sqrt{D})} \]

\[ \frac{c}{d} \geq 1 \quad \& \quad \sqrt{D} > 1 \quad \text{so} \quad \left| \frac{c}{d} - \sqrt{D} \right| < \frac{1}{2d^2} \]

We conclude \( \frac{c}{d} \) is a convergent of \( \sqrt{D} \).

**Example:** \( x^2 - 34y^2 = -1 \)

\( \sqrt{34} = \langle 5, 1, 4, 1, 10 \rangle \)
Convergents: \(
\{ \frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{16}{3}, \frac{27}{5}, \frac{43}{8}, \frac{67}{13}, \frac{110}{21}, \frac{177}{34}, \frac{287}{55}, \frac{464}{89}, \frac{751}{144} \} \)

\[
x^2 - 34y^2 = -3 \quad 2 \quad -9 \quad 1 \quad -9 \quad 2 \quad -9 \quad 1 \quad ...
\]

For a solution to \( x^2 - Dy^2 = \pm N \) we get

\[
\frac{c}{d} - \sqrt{D} = \frac{N}{d^2 \left( \frac{c}{d} + \sqrt{D} \right)}
\]

\[
\left( \frac{c}{d} \right)^2 = D + \frac{N}{d^2} \quad \text{so} \quad \frac{c}{d} > \sqrt{D}, \quad \frac{c}{d} + \sqrt{D} > 2\sqrt{D}
\]

So if \(|N| < \sqrt{D}\) the solution is a convergent of \(\sqrt{D}\).