REFERENCES: Humphreys II.8, Helgason III.3-III.4.
Recall, $\mathfrak{g}$ a semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a CSA. The root decomposition is $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$ where $\Delta \subseteq \mathfrak{h}^*$ are the roots, and since $\mathfrak{h}$ is ad-semisimple, the root spaces satisfy $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$.

**Theorem (1)**

1. $\dim \mathfrak{g}_{\alpha} = 1$ for all $\alpha \in \Delta$
2. If $\alpha, \beta \in \Delta$, $\alpha + \beta \neq 0$ then $\mathfrak{g}_{\alpha} \perp \mathfrak{g}_{\beta}$ (with respect to $B$)
3. $B|_{\mathfrak{h}}$ is nondegenerate. For each $\alpha \in \Delta$ there is a unique $H_{\alpha} \in \mathfrak{h}$ such that $B(H, H_{\alpha}) = \alpha(H)$.
4. If $\alpha \in \Delta$ then $-\alpha \in \Delta$, $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_{\alpha}$ and $\alpha(H_{\alpha}) \neq 0$. 
(2) is easy. For (3), if $H \in \text{rad } B|_{\mathfrak{h}}$, note that

$$B(H, g_\alpha) = 0 \text{ for all } \alpha \in \Delta$$

so $H \in \text{rad } B|_{\mathfrak{g}}$. Thus $H = 0$.

For (4), note that

$$\mathfrak{h} = \sum_{\alpha \in \Delta} [g_\alpha, g_{-\alpha}].$$

Let $X_\alpha \in g_\alpha$. $X_{-\alpha} \in g_{-\alpha}$. Then

$$B([X_\alpha, X_{-\alpha}], H) = B(X_\alpha, [X_{-\alpha}, H]) = \alpha(H)B(X_\alpha, X_{-\alpha}). \quad (1)$$

Thus $[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})H'_\alpha$ for some element $H'_\alpha \in \mathfrak{h}$. If we normalize the vectors so that $B(X_\alpha, X_{-\alpha}) = 1$, then because of (3), $[X_\alpha, X_{-\alpha}] = H_\alpha$ with the desired properties. \[\square\]
Proof of the theorem (1)

Note from a previous proof that for any root $\beta$ we have

$$\beta(H_\alpha) = r_\beta \alpha(H_\alpha) \quad r_\beta \in \mathbb{Q}$$

Then since

$$B(H_1, H_2) = \sum_{\gamma \in \Delta} \gamma(H_1) \gamma(H_2)$$

we get

$$B(H_\alpha, H) = \alpha(H_\alpha) \sum_{\beta \in \Delta} r_\beta \beta(H).$$

If $\alpha(H_\alpha) = 0$, $B(H_\alpha, H) = 0$ for all $H \in \mathfrak{h}$, so $H_\alpha = 0$. But $B(H_\alpha, H) = \alpha(H)$ which is not identically zero. So $\alpha(H_\alpha) \neq 0$. This implies that

$$\{ X_\alpha, H_\alpha := \frac{2}{\alpha(H_\alpha)} H_\alpha, X_{-\alpha} \}$$

forms an $sl(2, \mathbb{C})$. Write $\{ X_\alpha, H_\alpha, X_{-\alpha} \} = sl(2)_{\alpha}$. 
Proof of the theorem (1)

Let $\beta \in \Delta$ and consider the space

$$
\sum_{n \in \mathbb{Z}} g_{\beta + n\alpha}.
$$

(1)

This is an $sl(2)_\alpha$ module. It follows that $H_\alpha$ has to act by integers on $g_{\beta + n\alpha}$:

$$(\beta + n\alpha)(H_\alpha) \in \mathbb{Z}.$$  

(2)

So we get

$$
\frac{2B(H_\beta, H_\alpha)}{B(H_\alpha, H_\alpha)} \in \mathbb{Z}.
$$

Now let $q \leq n \leq p$ be the integers so that $g_{\beta + p\alpha + \alpha} = 0$, $g_{\beta + q\alpha - \alpha} = 0$. Then

$$(\beta + q\alpha)(H_\alpha) = - (\beta + p\alpha)(H_\alpha).$$

(3)

$$(\beta, \check{\alpha}) + 2q = -(\beta, \check{\alpha}) - 2p.$$  

(4)

So

$$p + q = - \frac{2B(H_\alpha, H_\beta)}{B(H_\alpha, H_\alpha)}.$$  

(5)

$B(H_\check{\alpha}, H_\beta)$ is always an integer.
Proof of the theorem (1)

Suppose $D_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is linearly independent of $X_{-\alpha}$. We assume as we may, $D_{-\alpha} \neq 0$ and $B(D_{-\alpha}, X_{+\alpha}) = 0$. Then $[D_{-\alpha}, X_{+\alpha}] = 0$ by the reason given right after (1) in the proof of theorem (1). So $D_{-\alpha}$ is a null vector for ad $X_{\alpha}$ while $[H_{\alpha}, D_{-\alpha}] = -2D_{-\alpha}$.

This cannot happen in an $sl(2)$–module. Thus $\dim \mathfrak{g}_{\alpha} = 1$ for all $\alpha \in \Delta$. In particular $\sum \mathfrak{g}_{\beta + n\alpha}$ is an irreducible $sl(2)_\alpha$–module.  \qed
Theorem (2)

1. The only roots proportional to $\alpha$ are $\pm \alpha$ and 0.
2. $[g_\alpha, g_\beta] = g_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$. 
Proof.

(1) Suppose $r \cdot \alpha$ with $r \neq 0$ is a root. Then $H_{r\alpha} = rH_{\alpha}$ and we find both $B(H_{\alpha}, H_{r\bar{\alpha}})$ and $B(H_{\bar{\alpha}}, H_{r\alpha})$ integers,

$$\frac{2B(H_{\alpha}, H_{\alpha}) \cdot r}{B(H_{\alpha}, H_{\alpha})} \in \mathbb{Z}, \quad \frac{2B(H_{\alpha}, H_{\alpha})r}{r^2 B(H_{\alpha}, H_{\alpha})} \in \mathbb{Z}.$$ 

The only nonzero choices for $r$ are $\pm \frac{1}{2}, \pm 1, \pm 2$. So suppose $\alpha$ is a root so that $2\alpha$ is a root as well. Let $X_{\pm\alpha}$ and $X_{\pm 2\alpha}$ be the root vectors. Since $3\alpha$ is not a root, $[X_{\alpha}, X_{2\alpha}] = 0$. Meanwhile $[X_{-\alpha}, X_{2\alpha}] = cX_{\alpha}$ because $\dim g_{\alpha} = 1$. The relation

$$0 = [X_{\alpha}, [X_{-\alpha}, X_{2\alpha}]] = [H_{\alpha}, X_{2\alpha}] + [X_{-\alpha}, [X_{\alpha}, X_{2\alpha}]] = 2\alpha(H_{\alpha})X_{2\alpha}$$

contradicts $\alpha(H_{\alpha}) \neq 0$.

(2) follows from the fact that $\sum g_{\alpha + n\beta}$ is an irreducible $sl(2)_{\alpha}$-module.
Theorem (3)

Let \( h_R := \sum_{\alpha \in \Delta} R H_\alpha \). Then

1. \( B|_h_R \) is real positive definite.
2. \( h = h_R + i h_R \).
Proof.

For (1), we compute $\alpha(H_\alpha)$:

$$\alpha(H_\alpha) = B(H_\alpha, H_\alpha) = \sum_{\beta \in \Delta} \beta(H_\alpha) \cdot \beta(H_\alpha)$$  \hspace{1cm} (1)

$$= \sum_{\beta \in \Delta} \frac{1}{2} \alpha(H_\alpha) \beta(H_\beta) \cdot \frac{1}{2} \alpha(H_\alpha) \beta(H_\beta) = \frac{\alpha(H_\alpha)^2}{4} \sum_{\beta \in \Delta} \beta(H_\beta)^2$$  \hspace{1cm} (2)

so $\alpha(H_\alpha)$ is nonnegative. Since it is nonzero, it is positive. (2) follows easily.
Theorem (4, Weyl’s normal form)

Let \( \mathfrak{h} \subseteq \mathfrak{g} \) be a CSA. There is a basis \( X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Delta \) such that

\[
[X_\alpha, X_{-\alpha}] = H_\alpha, \quad [H, X_\alpha] = \alpha(H)X_\alpha, \quad [X_\alpha, X_\beta] = N_{\alpha,\beta}
\]

satisfying

\[
N_{\alpha,\beta} = 0 \text{ if } \alpha + \beta \notin \Delta
\]

\[
N_{\alpha,\beta} = -N_{-\alpha,-\beta}.
\]

Furthermore,

\[
N_{\alpha,\beta}^2 = \frac{q(1 - p)}{2} \alpha(H_\alpha)
\]

where \( \beta + n\alpha, p \leq n \leq q \) is the \( \alpha \)-string through \( \beta \).

A proof can be found in the texts of Helgason, Jacobson, or Samelson.
A real vector space $V$ is said to have a complex structure if there is an $\mathbb{R}$-linear $J : V \rightarrow V$ such that $J^2 = -Id$. Then $V$ is a complex vector space with scalar multiplication

$$(a + ib)v = av + bJv.$$ 

Conversely if $E$ is a complex vector space, it is also a real vector space with complex structure via $J = i Id$.

**Definition**

A real algebra $\mathfrak{g}$ is said to have a complex structure if there is a complex structure $J$ such that $\text{ad} X \circ J = J \circ \text{ad} X$. For a complex algebra, a **real form** is a subalgebra $\mathfrak{g}_R \subset \mathfrak{g}$ satisfying $\mathfrak{g}_R \cap i\mathfrak{g}_R = (0)$ and $\mathfrak{g} = \mathfrak{g}_R + i\mathfrak{g}_R$.

Recall also that if $\mathfrak{g}$ is a real Lie algebra, its complexification is defined as $\mathfrak{g}_C := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with the obvious bracket structure. Then $\mathfrak{g}$ is a real form of $\mathfrak{g}_C$. 
Exercise

Show that a real Lie algebra $\mathfrak{g}$ is semisimple, solvable, nilpotent if and only if $\mathfrak{g}_c$ is semisimple, solvable, nilpotent.
Definition

A real Lie algebra \( \mathfrak{g} \) is called compact if the Cartan-Killing form \( B \) is negative definite.

Let \( \mathfrak{g} \) be complex semisimple and \( \Delta \) be the roots. A subset \( \Delta^+ \) is called a positive system if

1. \( \alpha, \beta \in \Delta^+ \), then \( \alpha + \beta \in \Delta^+ \) or else is not a root,
2. \( \Delta^+ \cup (-\Delta^+) = \Delta \).

Such systems always exist; choose an \( H_0 \in h_{\mathbb{R}} \) such that \( \alpha(H_0) \neq 0 \) for any \( \alpha \in \Delta \).

\( \Delta^+ := \{ \alpha : \alpha(H_0) > 0 \} \) is a positive system.

Conversely any positive system is given by this procedure, but this will be proved later.
Theorem (5)

Every semisimple Lie algebra $\mathfrak{g}$ has a compact real form.

Proof.

Choose a positive system $\Delta^+$ and consider the subspace

$$\mathfrak{g}_k = i\mathfrak{h}_\mathbb{R} + \sum_{\alpha \in \Delta^+} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta^+} \mathbb{R}(X_\alpha + iX_{-\alpha})$$

where the root vectors are in normal form as in theorem 4. It is not hard to see that $B$ is negative definite when restricted to this subspace; the vectors $X_\alpha$ have to satisfy $B(X_\alpha, X_{-\alpha}) = 1$ since $[X_\alpha, X_{-\alpha}] = H_\alpha$. The fact that it is a subalgebra also follows from theorem 4.
Recall

\[ \text{ad}: \mathfrak{g} \to \text{Der}(\mathfrak{g}). \]  

(3)

For \( \mathfrak{g} \) semisimple this is an isomorphism. Recall

\[ \text{Aut}(\mathfrak{g}) := \{ A \in \text{GL}(\mathfrak{g}) \mid A([x, y]) = [Ax, Ay] \}. \]  

(4)

\( \text{Int}(\mathfrak{g}) := \) the closure of the subgroup of \( \text{Aut}(\mathfrak{g}) \) generated by \( e^{\text{ad}x} \) with \( x \in \mathfrak{g} \).

The Lie algebras of these groups are \( \text{Der}(\mathfrak{g}) \) and \( \text{ad}\mathfrak{g} \subset \text{Der}(\mathfrak{g}) \).

\( \text{Int}(\mathfrak{g}) \) is also called the \textit{adjoint group} of \( \mathfrak{g} \).

They are both equal to \( \mathfrak{g} \) if \( \mathfrak{g} \) is semisimple. Thus in this case, \( \text{Int}(\mathfrak{g}) \) is the connected component of the identity of \( \text{Aut}(\mathfrak{g}) \).

Note that for \( A \in \text{Aut}(\mathfrak{g}) \), \( \text{ad} Ax = A \circ \text{ad} x \circ A^{-1} \) because

\[ \text{ad}(Ax)(y) = [Ax, y] = A([x, A^{-1}y]) = A \circ \text{ad} x \circ A^{-1}(y). \]

Then

\[ B(Ax, Ay) = \text{Tr}(\text{ad} Ax \circ \text{ad} Ay) = \text{Tr}(A \circ \text{ad} x \circ A^{-1} \circ A \circ \text{ad} y \circ A^{-1}) = \]

\[ = \text{Tr}(\text{ad} x \circ \text{ad} y). \]

(5)
Theorem (6)

If $B$ is negative definite, $\text{Aut}(\mathfrak{g})$ and $\text{Int}(\mathfrak{g})$ are closed subgroups of an orthogonal group, therefore compact.

Proof.

From the above discussion, $\text{Aut}(\mathfrak{g}) \subset O(B)$, the group leaving $B$ invariant. When $B$ is negative definite, this group is compact.
Theorem

Suppose $G$ is a compact Hausdorff topological group. Then $G$ has a unique biinvariant (Borel) measure.

\[ f \in C_c(G) \mapsto \int_G f(x) d\mu(x) \quad (6) \]

such that

\[ \int_G f(gx) d\mu(x) = \int_G f(xg) d\mu(x) = \int_G f(x) d\mu(x) \quad (7) \]

for all $g \in G$ and $f \in C_c(G)$. 

Proposition

Suppose $(\pi, V)$ is a representation of a compact group $G$. $V$ a finite dimensional complex vector space. Then $V$ has a $G$–invariant inner product.

Proof.

Let $\langle \ , \ \rangle$ be any inner product. Define

$$ (v, w) = \int_G \langle \pi(x)v, \pi(x)w \rangle d\mu(x). \quad (8) $$
Exercise

*Complete the proof.*

Corollary

*Any finite dimensional representation of $G$ is completely reducible.*

Proof.

Let $\langle \ , \ \rangle$ be a $G$–invariant inner product. If $W \subseteq V$ is an invariant subspace, then $W \perp$ is also $G$–invariant.
**Theorem**

Let \((\pi, V)\) be a finite dimensional representation of a complex semisimple Lie algebra. Then \((\pi, V)\) is completely reducible.

**Proof.**

Let \(g_R\) be a compact real form. A nontrivial result asserts that there is a simply connected Lie group \(G_R\) with Lie algebra \(g_R\). Standard properties of Lie groups imply that \((\pi, V)\) exponentiates to a representation of \(G_R\). Let \(W\) be a \(g\)–invariant subspace. It is \(G_R\)–invariant, so it has a \(G_R\)–invariant complement, \(W'\). Then \(W'\) is \(g_R\) invariant. Since \(g = g_R \otimes \mathbb{R} \mathbb{C}\), and \(W'\) is complex, \(W'\) is \(g\) invariant.

**References:** F. Warner, Hausner-Schwartz.