Levi’s Theorem

Let \( \mathfrak{g} \) be an arbitrary Lie algebra and \( \mathfrak{a} \) an ideal such that the quotient \( \mathfrak{s} \) is semisimple. Then \( \mathfrak{a} \) has a complement which is a Lie algebra.

We can rephrase this as follows. Suppose

\[
0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{s} \longrightarrow 0
\]  

is an exact sequence of Lie algebras such that \( \mathfrak{s} \) is semisimple. Then there is a Lie algebra homomorphism \( \epsilon : \mathfrak{s} \longrightarrow \mathfrak{g} \) such that \( \pi \circ \epsilon = Id \).
Proof of Levi’s theorem

Proof.

Suppose \( \mathfrak{a} \) has a nontrivial proper ideal \( \mathfrak{a}_1 \). Then there is an exact sequence

\[
0 \rightarrow \mathfrak{a}/\mathfrak{a}_1 \rightarrow \mathfrak{g}/\mathfrak{a}_1 \xrightarrow{\pi} \mathfrak{s} \rightarrow 0
\]  

(2)

By induction on the dimension of \( \mathfrak{a} \), we can assume that \( \mathfrak{a}/\mathfrak{a}_1 \) has a complement \( \mathfrak{s}_1 \). Then there is an exact sequence

\[
0 \rightarrow \mathfrak{a}_1 \rightarrow \mathfrak{g}_1 \xrightarrow{\pi} \mathfrak{s} \rightarrow 0
\]  

(3)

where \( \mathfrak{g}_1 \) is the inverse image of \( \mathfrak{s}_1 \). By induction on the dimension of \( \mathfrak{g} \), there is a Lie algebra complement \( \mathcal{V} \) to \( \mathfrak{a}_1 \) in \( \mathfrak{g}_1 \). Then a dimension count shows that this is a complement to \( \mathfrak{a} \) in \( \mathfrak{g} \) as well.
Thus we may assume that $\mathfrak{a}$ has no proper ideals. Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$. Then its image under $\pi$ is a solvable ideal in $\mathfrak{s}$, therefore $0$. In other words $\mathfrak{r} \subseteq \mathfrak{a}$. If $\mathfrak{r} = (0)$, the whole algebra $\mathfrak{g}$ is semisimple and we’re done. If $\mathfrak{r} = \mathfrak{a}$, then $\mathfrak{a}$ is solvable, so $[\mathfrak{a}, \mathfrak{a}]$ is a proper ideal, therefore zero. Thus $\mathfrak{a}$ is abelian. Then the adjoint action of $\mathfrak{g}$ factors through $\mathfrak{a}$, so $\mathfrak{a}$ is a module for $\mathfrak{s}$. The fact that $\mathfrak{a}$ has no proper ideals means that this module is simple. If this module were trivial, then $\mathfrak{a}$ would be contained in the center of $\mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{a} = \mathfrak{s}$ acts on all of $\mathfrak{g}$ via the adjoint action. Since $\mathfrak{s}$ is semisimple, the representation is completely reducible and the claim again follows.
Proof of Levi’s theorem

continued.

Thus we may assume we are in the case when $\mathfrak{a}$ is abelian and an irreducible nontrivial module for $\mathfrak{s}$. We can write $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ as a vector space. The bracket must take the form

$$
[(a_1, x_1), (a_2, x_2)] =
= ([a_1, a_2] + \tau(x_1)a_2 - \tau(x_2)a_1 + \phi(x_1, x_2), [x_1, x_2]) = (4)
= (\tau(x_1)a_2 - \tau(x_2)a_1 + \phi(x_1, x_2), [x_1, x_2]).
$$

The term $[a_1, a_2]$ drops out because we assumed $\mathfrak{a}$ abelian. $\tau : \mathfrak{s} \rightarrow \text{End}(\mathfrak{a})$ is given by $\tau(x)a := [(a, 0), (0, x)]$, which must be in $\mathfrak{a}$, and $\phi : \Lambda^2\mathfrak{s} \rightarrow \mathfrak{a}$ is the component of $[(0, x_1), (0, x_2)]$ in $\mathfrak{a}$. The Jacobi identity implies that $\phi$ is a cocycle.
Two such realizations are equivalent if there is an isomorphism
\[ \Psi : \mathfrak{a} \oplus \mathfrak{s} \longrightarrow \mathfrak{a} \oplus \mathfrak{s} \]
of the form \[ \Psi(a, x) = (a + \sigma(x), x) \]
satisfying \[ \Psi([[a_1, x_1], (a_2, x_2)])_1 = [\Psi(a_1, x_1), \Psi(a_2, x_2)]_2. \]
This is equivalent to \[ \phi_1(x_1, x_2) - \phi_2(x_1, x_2) = \sigma([x_1, x_2]), \]
i.e. \[ \phi_1 - \phi_2 \] is a coboundary. But because \( \mathfrak{s} \) is semisimple,
\[ H^2(\mathfrak{s}, \mathfrak{a}) = H^2(\mathfrak{s}) \otimes \mathfrak{a}^s = \Lambda^2(\mathfrak{s})^s \oplus \mathfrak{a}^s = 0 \quad (5) \]
since \( \mathfrak{a} \) was assumed nontrivial.

In fact \( \Lambda^2(\mathfrak{g})^\mathfrak{g} = 0 \) for any semisimple Lie algebra \( \mathfrak{g} \). \( \square \)
Suppose $\pi : \mathfrak{s} \longrightarrow \text{Der}(\mathfrak{a})$ is a representation, where $\mathfrak{s}$ and $\mathfrak{a}$ are Lie algebras. Then we can form a new Lie algebra $\mathfrak{g} := \mathfrak{s} \ltimes \mathfrak{a}$ as follows. The space of $\mathfrak{g}$ is $\mathfrak{s} \times \mathfrak{a}$. The bracket is

$$[[x_1, a_1], (x_2, a_2)] = ([x_1, x_2], [a_1, a_2] + \pi(x_1)(a_2) - \pi(x_2)(a_1)).$$

(6)

This algebra is called the semidirect product of $\mathfrak{s}$ and $\mathfrak{a}$, and $\mathfrak{a}$ is an ideal in $\mathfrak{g}$.

**Corollary (Levi’s theorem)**

*Every Lie algebra is the semidirect product of its solvable radical with a semisimple algebra.*

The semisimple algebra is called the *Levi component* of $\mathfrak{g}$. The corollary is a special case of the theorem, $\mathfrak{a} = \mathfrak{r}(\mathfrak{g})$. 
The radical is unique, but the Levi component is not. It is unique up to conjugation by inner automorphisms (elements of the connected Lie group in $Aut(\mathfrak{g})$ whose Lie algebra is the algebra of derivations of $\mathfrak{g}$.) The proof is very similar to the analogous statement about the conjugacy of all Cartan subalgebras. It is called the Malcev-Harish-Chandra theorem.

Suppose $S$ and $A$ are groups, and $\Pi : S \to Aut(A)$ is a group homomorphism. We can form the semidirect product $A \ltimes S$ as follows. The space is $S \times A$, and the product is

$$(s_1, a_1) \cdot (s_2, a_2) = (s_1 s_2, \Pi(s_2^{-1})a_1 a_2). \quad (7)$$

Then $\pi := d\Pi : S \to Der(\mathfrak{a})$ defines a semidirect product $s \ltimes \mathfrak{a}$, which is the Lie algebra of $S \ltimes A$. 
Proposition

For every Lie algebra $\mathfrak{g}$ there is a Lie group $G$ such that $\text{Lie}(G) = \mathfrak{g}$.

Proof.

Levi’s theorem reduces the proof of the proposition to the case when $\mathfrak{g}$ is solvable. Then there is a sequence of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n \supset \mathfrak{g}_{n+1} = (0) \quad (8)$$

such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) = 1$. Choose $\nu \in \mathfrak{g}/\mathfrak{g}_1$. Then $\mathfrak{h}_0 := \mathbb{K}\nu$ is a subalgebra, and we can write $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{h}_0$ with $\tau$ given by the action of $\text{ad} \nu$ on $\mathfrak{g}_1$.

The proof follows by induction, and using the construction of semidirect products of groups and Lie algebras.
This proposition is made clearer by the following theorem.

**Theorem (Ado)**

*Every Lie algebra has a representation* $\pi : g \rightarrow \text{End}(V)$ *which is faithful, i.e.* $\ker \pi = (0)$.  