1. Universal Enveloping Algebra

References: Dixmier, Enveloping Algebras, Serre Lie Algebras

1.1. An associative algebra $\mathcal{U}(\mathfrak{g})$ with unit together with a Lie algebra homomorphism $\varepsilon : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is called a universal enveloping algebra if for every Lie algebra map
\[
\phi : \mathfrak{g} \to A
\]
where $A$ is an associative algebra with unit, there is a unique extension
\[
\varepsilon : \mathfrak{g} \overset{\phi}{\rightarrow} A \overset{\rho}{\leftarrow} \mathcal{U}(\mathfrak{g})
\]
that makes the diagram commutative and satisfying $\rho(1) = 1$.

Lemma. If $(\varepsilon, \mathcal{U}(\mathfrak{g}))$ exists, it is unique.

Proof. If there were two such objects $(\varepsilon_1, \mathcal{U}_1)$ and $(\varepsilon_2, \mathcal{U}_2)$, we would have maps $\rho_1 : \mathcal{U}_1 \to \mathcal{U}_2$ and $\rho_2 : \mathcal{U}_2 \to \mathcal{U}_1$ which would make the diagrams corresponding to (1.1.1) commute. But then the uniqueness implies $\rho_1 \circ \rho_2 = id, \rho_2 \circ \rho_1 = id$. \qed

1.2. Existence. Form
\[
T\mathfrak{g} := C \oplus T^1(\mathfrak{g}) \oplus T^2(\mathfrak{g}) \oplus \cdots,
\]
(recall $T^0(\mathfrak{g}) = C$ and $T^1(\mathfrak{g}) = \mathfrak{g}$). This is the “universal associative algebra”; if
\[
\varphi = \mathfrak{g} \to A
\]
is any linear map, it extends uniquely to an algebra homomorphism
\[
\Phi : T(\mathfrak{g}) \to A
\]
such that $\Phi(\mathbb{1}) = \mathbb{1}$. Let $J$ be the two sided ideal generated by
\[
x \otimes y - y \otimes x - [x, y],
\]
and define $\mathcal{U}(\mathfrak{g}) : T(\mathfrak{g})/J$. If $\varphi$ is a Lie algebra homomorphism, then $\Phi$ factors to an algebra homomorphism
\[
\rho : \mathcal{U}(\mathfrak{g}) \to A, \quad \rho(\mathbb{1}) = \mathbb{1}.
\]
Since $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$, $\rho$ is unique.

Lemma. The canonical map from $C$ to $\mathcal{U}(\mathfrak{g})$ is an injection.
Remark: If \( g \) is the Lie algebra of a Lie group, then \( \mathcal{U}(g) \) identifies with left invariant differential operators.

Example: If \( g \) is abelian, then \( \mathcal{U}(g) \) is the symmetric algebra \( S(g) \),
\[
S(g) := T(g)/\{x \otimes y - y \otimes x\}.
\]
Note that \( S(g) \) is graded because \( T(g) \) is graded, \( T(g) = \oplus T^n(g) \)
\[
T^n(g) \cdot T^m(g) \subseteq T^{n+m}(g)
\]
and \( J \) is generated by homogeneous elements of order 2.

In general this is not the case; \( \mathcal{U}(g) \) only admits a filtration
\[
\mathcal{U}_n(g) := \text{image of } T^n(g) \text{ with } i \leq n.
\]
Then
\[
\begin{align*}
(1) \quad & \mathcal{U}_n(g) \cdot \mathcal{U}_m(g) \subseteq \mathcal{U}_{n+m}(g) \\
(2) \quad & \text{if } a \in \mathcal{U}_n, \ b \in \mathcal{U}_m, \ [a, b] := ab - ba \in \mathcal{U}_{n+m-1}.
\end{align*}
\]

In such a case one defines the graded object,
\[
Gr \mathcal{U}(g) := \oplus \mathcal{U}_n/\mathcal{U}_{n-1}.
\]
Then \( Gr \mathcal{U}(g) \) is a graded algebra and (2) implies it is commutative. In addition there is a well-defined map
\[
\Phi : \mathcal{U}(g) \to Gr \mathcal{U}(g) \quad \text{(also written } x \mapsto \overline{x})
\]
This is not an algebra map. Now consider \( g_{ab} \), the algebra with the same underlying space but trivial bracket. The composition map
\[
g_{ab} \to g \to \mathcal{U}(g) \to Gr \mathcal{U}(g)
\]
is a Lie algebra map, so we get an algebra map
\[
\psi : S(g) \to Gr \mathcal{U}(g).
\]

Theorem (Poincaré-Birkhoff-Witt, PBW). \( \psi \) is an isomorphism. Actually we can define a map
\[
\tilde{\psi} : S(g) \to \mathcal{U}(g)
\]
by the formula
\[
x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.
\]
The claim is equivalent to the fact that \( \tilde{\psi} \) is an isomorphism.

Exercise 1. (1) Check that \( \tilde{\psi} \) is well defined.
(2) Check that \( \psi = \Phi \circ \tilde{\psi} \).

Proof. Theorem (1.2) can be rephrased as follows. Let \( \{e_i\}_{i \in I} \) be a basis for \( g \). Choose a total ordering on \( I \). If \( M = (i_1, \ldots, i_n) \) with \( i_j \leq i_{j+1} \) is an \( n \)-tuple, let
\[
e_M := e_{i_1} \cdots e_{i_n} \in \mathcal{U}_n \quad (e_{\emptyset} = 1).
\]
Then theorem 1.2 is equivalent to

\[(1.2.1) \quad \{e_M\} \text{ is a } k\text{-basis for } \mathcal{U}(\mathfrak{g}).\]

We need to show that if \(\sum c_M e_M = 0\) then all \(c_M = 0\).

We define a representation of \(\mathfrak{g}\) on \(S(\mathfrak{g})\) as follows. For \(M\) as before, let \(x_M = e_{i_1} \cdots e_{i_n} \in S(\mathfrak{g})_n \ (x_\emptyset = \mathds{1})\). It is clear that these form a basis. Then define \(\pi(e_i)\) by induction on \(|M|\), the number of terms in \(M\).

1. \(\pi(e_i)x_\emptyset = x_{(i)}\).

2. Suppose \(|M| = n - 1\). If \(i \leq i_1\), then set
   \[\pi(e_i)x_M = x_{(i,i_1,...,i_{n-1})}\]
   If \(i > i_1\), then define
   \[\pi(e_i)x_M = \pi(e_{i_1})\cdot (\pi(e_i)x_{(i_2,...,i_n)}) + \pi([e_i,e_{i_1}]) \cdot x_{(i_2,...,i_n)}\]

It is "clear" that this is a Lie algebra homomorphism into \(\text{End}(S(\mathfrak{g}))\). Thus we get a Lie homomorphism

\[\pi : \mathcal{U}(\mathfrak{g}) \to \text{End}(S(\mathfrak{g})).\]

If \(\sum c_M e_M = 0\), then

\[0 = \sum c_M \pi(e_M)x_\emptyset = \sum c_M x_M \Rightarrow c_M = 0.\]

\[\square\]

1.3. Note that \(\mathfrak{g}\) acts on itself by \(\text{ad} : \)

\[\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}).\]

This extends to an action on \(T(\mathfrak{g})\)

\[\text{ad}_x(x_1 \otimes \cdots \otimes x_n) := \sum_{i=1}^n x_1 \otimes \cdots \otimes \text{ad}_x(x_i) \otimes \cdots \otimes x_n.\]

Then the ideals generated by

\[I := \{x \otimes y - y \otimes x \} , \quad J := \{x \otimes y - y \otimes x - [x,y]\}\]

are invariant subspaces. So we get actions

\[\text{ad} : \mathfrak{g} \to \text{End}(\text{S}(\mathfrak{g})), \quad \mathfrak{g} \to \text{End}(\mathcal{U}(\mathfrak{g})).\]

**Exercise 2.** Show that the second action coincides with

\[\text{ad}_x(u) := xu - ux \quad \square\]

Since the action preserves the \(\mathcal{U}_n\), we also get

\[\text{ad} : \mathfrak{g} \to \text{End}(\text{Gr} \mathcal{U}(\mathfrak{g})).\]
Proposition. The maps
\[ \tilde{\psi} : S(g) \to \mathcal{U}(g), \quad \Phi : \mathcal{U}(g) \to Gr\mathcal{U}(g) \]
are intertwining maps, i.e.,
\[ \tilde{\psi} \circ ad_x = ad_x \circ \tilde{\psi}, \quad \Phi \circ ad_x = ad_x \circ \Phi \quad \forall x. \]

Proof. Clear. \qed

1.4. Some consequences.

I: If \( g \simeq g_1 \times g_2 \), then
\[ \mathcal{U}(g) \simeq \mathcal{U}(g_1) \otimes \mathcal{U}(g_2). \]

II: We can identify \( Gr\mathcal{U}(g) \) with \( S(g) \). The map \( g \to \mathcal{U}(g) \) is injective.

III: Suppose \( h, \mathfrak{t} \subseteq g \) are subalgebras such that \( g = h + \mathfrak{t} \). Let \( \ell = h \cap \mathfrak{t} \).
There exists a unique linear map
\[ \varphi : \mathcal{U}(h) \otimes \mathcal{U}(\ell) \mathcal{U}(\mathfrak{t}) \to \mathcal{U}(g), \quad \varphi(v \otimes w) = v \cdot w. \]
Here \( \mathcal{U}(h) \) is a left, \( \mathcal{U}(\mathfrak{t}) \) a right module for \( \mathcal{U}(\ell) \).

IV: There exists a unique antiautomorphism \( T \) of \( \mathcal{U}(g) \) such that \( x^T = -x \) for \( x \in g \) (\( 1^T = 1 \)).

V: If \( g \) is finite dimensional then \( \mathcal{U}(g) \) is noetherian.

VI: \( S(g)^\theta \simeq \mathcal{U}(g)^\theta \).

VII: The algebra \( \mathcal{U}(g) \) has a Hopf algebra structure.

Examples:

(a): \( g = sl(2) \). Any element in \( \mathcal{U}(g) \) can be written uniquely as
\[ \sum c_{i,j,k} f^i h^j e^k. \]
Let \( b \) be the subalgebra of upper triangular matrices, \( n \) the subalgebra of strictly upper triangular matrices and \( \bar{n} \) the subalgebra of strictly lower triangular matrices. Then
\[ \mathcal{U}(\bar{n}) \otimes \mathcal{U}(b) \to \mathcal{U}(g). \]

(b): \( g = \mathcal{H}_n \), the Heisenberg algebra with basis \( p_1 \cdots p_n, q_1 \cdots q_n, z \),
and Lie bracket \([p_i, p_j] = [q_i, q_j] = 0, [p_i, q_j] = \delta_{ij} z\). \( \mathcal{U}(g) \) has a basis
\( \{p^A q^B z^C\} \) where \( p^A = p_1^{i_1} \cdots p_n^{i_n} \) and so on.

1.5. Proof of (V). Let \( A = \bigoplus_{n=0}^{\infty} A_n \) be a graded noetherian ring (abelian or at least such that \( A_0 \) is central).

Lemma. (1) \( A_0 \) is noetherian.
(2) \( A \) is a finitely generated algebra.
Proof. For (1), let $A_+: = \bigoplus_{n=1}^\infty A_n$. Then $A_0 = A/A_+$, and the claim follows from standard ring theory. For (2), let $x_1, \ldots, x_s$ be homogeneous generators (for $A$ as a left $A$-module) and $d_i = \deg(x_i)$. Let $B$ be the $A_0$-subalgebra generated by the $x_i$. We claim that $A_n \subset A$ by induction. Indeed, $A_0 \subset B$. Let $y \in A_n$. Then $y = \sum y_i x_i$ and $\deg(y_i) = n - d_i < n$. By induction $y_i \in B$ so $y \in B$. □

Let now $D$ be an associative ring with identity and an increasing filtration $D_n$ of subspaces such that

1. $D_n = (0)$ for $n < 0$,
2. $\bigcup_{0 \leq n} D_n = D$,
3. $1 \in D_0$,
4. $D_n \cdot D_m \subset D_{n+m}$,
5. $[D_n, D_m] \subset D_{n+m-1}$,
6. $Gr(D)$ is noetherian,
7. $Gr_1(D)$ generates $Gr(D)$ as a $D_0$-algebra.

Let $A := Gr(D)$. By (4-6) it satisfies the conditions of the lemma. So $D_0$ is a noetherian ring and we also get

$$Gr_{n+1}(D) = Gr_1(D) \cdot Gr_n(D), \quad D_{n+1} = D_1 \cdot D_n.$$

**Proposition.** $D$ is a left and right noetherian ring.

**Proof.** Let $I$ be a left ideal. We need to show it is finitely generated. It has a filtration by $I_n := I \cap D_n$ which satisfies

$$D_n \cdot I_m \subset I_{n+m}, \quad I_n = (0) \text{ for } n < 0, \cup I_n = I.$$

Then $Gr(I)$ is an ideal in $Gr(D)$, so it must be finitely generated as an $A$-module. This implies the following for the filtration $I_n$

1. $I_n$ is finitely generated as a $D_0$-module,
2. there is $p > 0$ such that $D_n \cdot I_m = I_{n+m}$ for all $n > 0$ and all $m > p$.

These two properties imply that $I$ is finitely generated. □

**Remark:** This setup is motivated by the theory of rings of differential operators. The universal enveloping algebra is such a ring, the left invariant operators on a Lie group.

A filtration with these properties along with (1.5.1), is called a good filtration.

1.6.

**Lemma (Schur’s Lemma).** Suppose $(\pi, V)$ is an irreducible representation of an algebra $A$ (with $I$), such that $V$ has a countable basis. Then the center of $A$ acts by scalars.

**Proof.** Let $a \in A$ and $\lambda \in \mathbb{C}$. Then if $a - \lambda I$ has a nontrivial kernel, it is $g$-invariant so either equal to 0 or the whole space. If it is the whole space, done. On the other hand, if $Im(a - \lambda I)$ is not the whole space, it is a
nontrivial invariant subspace, and we are done again. So assume all \( a - \lambda I \)
are invertible. Let \( v \neq 0 \). Then
\[
\{(a - \lambda)^{-1}v\}_{\lambda \in \mathbb{C}}
\]
are linearly independent, a contradiction to the fact that the space \( V \) has a countable basis. Say
\[
\sum_{i=1}^{n} c_i (a - \lambda_i)^{-1}v = 0
\]
for some \( 0 \neq v \). Multiply by \( \Pi(a - \lambda_i) \). We find that there is a polynomial
\( p(t) \neq 0 \) such that \( p(a)v = 0 \). It follows (since \( p(t) = \Pi(t - \mu_j) \)) that some
\( (a - \mu_j)v = 0 \), a contradiction. \( \square \)

1.7. Note that \( g = \text{sl}(2) \) has trivial center but \( \mathcal{U}(g) \) does not. First recall that in general \( g \) acts on \( \mathcal{U}(g) \) via
\[
adx(x) := \sum u_1 \cdots adx(u_i) \cdots u_n = x \cdot u - u \cdot x.
\]
So the center of \( \mathcal{U} \) is the same as \( \mathcal{U}(g)^0 \), the elements centralized by the adjoint action of \( g \). \( g \) also acts on \( S(g) \) by the same formula
\[
x \cdot (v_1 \cdots v_n) := \sum_i v_1 \cdots adx(v_i) \cdots v_n.
\]

Exercise 3. 
(1) Prove property (VI) in section (1.4).
(2) Show that \( S(g)^0 \) is a polynomial algebra generated by \( ef + \frac{1}{4}h^2 \).
(3) Show that \( \mathcal{U}(g)^0 \) is generated by \( 2(ef + fe) + h^2 \) (use the map \( S(g) \to \mathcal{U}(g) \)). Check that this acts by a scalar on \( F(n) \). \( \square \)

1.8. The Weyl algebra. Let \( \mathcal{H}_n \) be the associative algebra generated by
\[
\{p_1, \ldots, p_n, q_1, \ldots, q_n\}
\]
\[
[p_i, p_j] = [q_i, q_j] = 0, \quad [p_i, q_j] = \delta_{ij}z.
\]
Let
\[
\mathcal{A}_n \simeq \mathcal{U}(\mathcal{H}_n)/(z - 1).
\]

Exercise 4. Show that \( \mathcal{A}_n \) is isomorphic to the algebra of differential operators on \( \mathbb{C}^n \) with polynomial coefficients.

Schur’s lemma implies that \( z \) must act by a scalar on any irreducible representation of \( \mathcal{H}_n \).

Proposition. Every two sided ideal of \( \mathcal{A}_n \) is either 0 or \( \mathcal{A}_n \). We say \( \mathcal{A}_n \) is simple.

Corollary. If \( \varphi : \mathcal{A}_n \to \mathcal{A}_n \) is an algebra homomorphism, then \( \varphi \) is injective.
1.9. Relation to PDE’s. To a system of equations $S := \{P_i f = 0\}$ for some $P_i \in \mathcal{A}_n$ we can associate the left module $M := \mathcal{A}_n / \{\sum A_n P_i\}$. Then

$$\text{Hom}_{\mathcal{A}_n}[M, \mathbb{C}[x_1, \ldots, x_n]] \simeq \text{Sol}(S)$$

as vector spaces. Here

$$\text{Sol}(S) := \{f \in \mathbb{C}[x_1, \ldots, x_n] : P_i f = 0\}.$$

Examples:

1. $P_i = x_i$. $\mathcal{M} = \mathcal{A}_n / \sum A_n x_i \simeq \mathbb{C}[\partial_1 \cdots \partial_n]$. The solution to this is the delta function, not in $\text{Sol}(S)$ which is 0. But still it is important to consider such solutions.

2. $P_i = \partial_i$. $\mathcal{M} = \mathcal{A}_n / \sum A_n \partial_i \simeq \mathbb{C}[x_1 \cdots x_n]$. $\text{Sol}(S) = \mathbb{C}1$.

3. The Fourier transform can be defined algebraically on $\mathcal{A}_n$ by the formulas

$$x_i \mapsto -\sqrt{-1} \partial_i$$
$$\partial_i \mapsto +\sqrt{-1} x_i.$$

It is an isomorphism of $\mathcal{A}_n$ which interchanges examples (1) and (2).

1.10. Jacobian Conjecture. If

$$F = (F_1, \ldots, F_n) : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$$

is such that $\Delta F := \det \left( \frac{\partial F_i}{\partial x_j} \right) \equiv 1$, then $F$ has a polynomial inverse. □

Such an $F$ gives rise to a

$$\varphi_F : \mathcal{A}_n \to \mathcal{A}_n.$$

For $(f_1, \ldots, f_n)$, write

$$J(f_1, \ldots, f_n) := \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\
\end{pmatrix}.$$

Then the definition is in the next exercise.

Exercise 5. Show that the assignments

$$x_i \mapsto F_i$$
$$\partial_i : f \mapsto \det J(F_1 \cdots f \cdots F_n).$$

define a homomorphism $\varphi : \mathcal{A}_n \to \mathcal{A}_n$.

Conjecture. Any nonzero homomorphism is onto. This conjecture implies the Jacobian conjecture (cf. Coutinho, a primer of algebraic D-modules).
1.11. Fundamental Solutions of PDE with Constant Coefficients.
Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be such that $p(x) \geq 0$ for all $x$. We can define a distribution

$$T_\lambda(f) := \int_{\mathbb{R}^n} p(x)^\lambda f(x) dx.$$ 

This is tempered for $\Re \lambda \geq 0$. Assume that $T_\lambda$ extends meromorphically in $\lambda$ in the complex plane. At $\lambda = -1$

$$T_\lambda(f) = \sum_{n \geq -N} (\lambda + 1)^n S_n(f).$$

Then $p \cdot T_\lambda(f) := T_\lambda(pf) = T_{\lambda+1}(f)$ is well defined at $\lambda = 0$ and equals 1. Thus

$$p \cdot S_n = 0 \quad n < 0$$
$$pS_0 = 1.$$  

If $p \in \mathbb{C}[x_1, \ldots, x_n]$, let $q = p \cdot \bar{p}$. Then $q \cdot S_0 = 1$ implies $p \cdot (\bar{p}S_0) = 1$. Taking Fourier transforms

$$\partial_\rho S_0 = \delta_0, \quad \bar{p}(x) = p(-\sqrt{-1}x)$$

Say $p(x) = x_1^2 + \cdots + x_n^2$. Then one shows that $T_\lambda$ extends meromorphically with poles at $-1, -2 \cdots$. (cf. Gelfand-Shilov, Generalized functions volume 1).

**Theorem** (Bernstein). Let $p \in \mathbb{C}[x_1, \ldots, x_n]$. There is $D(\lambda) \in A_n$ polynomial in $\lambda$ and $b(\lambda) \in \mathbb{C}[\lambda]$ such that

$$D(\lambda)p^{\lambda+1} = b(\lambda)p^\lambda.$$  

**Corollary.** $T_\lambda$ extends meromorphically with poles on finitely many arithmetic progressions $\{\lambda_j - k\}_{k \in \mathbb{N}}$.

**Proof.** The relation

$$T_\lambda(f) = \int p^\lambda f = \frac{1}{b(\lambda)} \int p^{\lambda+1} D(\lambda)^* f$$

follows from theorem (1.11). By iterating we can see that $T_\alpha$ can be expressed in terms of $p^{\lambda+n}$, which (as a distribution) is well defined and holomorphic in $\lambda$ for large enough $n$. The only poles can come from zeroes of the denominator which is a product of $b(\lambda + k)$. The claim follows. 

1.12. Some Remarks on Exercise 5. Recall $\mathfrak{g} \simeq sl(2, \mathbb{C})$. We are looking for $\mathfrak{g}$-invariants in $S(\mathfrak{g})$. First observe that we can realize $\mathfrak{g}$ as $2 \times 2$ matrices with trace 0.

**Definition.** [Dual Representation] If $(\pi, V)$ is a representation of $\mathfrak{g}$, we define $(\pi^*, V^*)$ where $V^*$ is the (linear) dual of $V$ and the action is

$$(\pi^*(x)f)(y) := -f(\pi(x)(y)).$$
It is easy to check that 
\[ \pi^*([a,b]) = [\pi^*(a), \pi^*(b)]. \]

**Lemma.** If \( V \simeq g \simeq sl(2) \) and \( \pi = ad \), then \( (\pi, V) \simeq (\pi^*, V^*) \).

**Proof.** The bilinear form 
\[ (x, y) \mapsto tr(xy) \]
is nondegenerate and satisfies 
\[ (\text{ad} a \cdot x, y) + (x, \text{ad} a \cdot y) = 0. \]

As a result, \( S(g) \simeq S(g^*) = P(g) \). Next observe that \( G = SL(2, \mathbb{C}) := \{2 \times 2 \text{ matrices with determinant 1}\} \) also acts on \( g \) (and therefore on \( S(g) \) and \( P(g) \)) by
\[ \text{Ad} : (g, x) \mapsto gxg^{-1}. \]
Then \( sl(2, \mathbb{C}) \) is the Lie algebra of \( SL(2, \mathbb{C}) \) and the action differentiates to
\[ \frac{d}{dt} \bigg|_{t=0} e^{tX} \cdot x \cdot e^{-tX} = Xx - xX = [X, x]. \]
In other words, the differential of \( \text{Ad} \) is \( \text{ad} \).

**Proposition.** \( S(g)^G = S(g)^g \). Clearly \( S(g)^G \subseteq S(g)^g \).

To show the converse,

**Lemma.** \( \exp : g \to G \) is a local diffeomorphism from an open set \( 0 \in U \subseteq g \) onto an open set \( 1 \in V \subseteq G \).

**Proof.** Exercise. □

**Hint:** The differential of \( \exp \) at 0 is \( I \). use the inverse function theorem. □

Let 
\[ h := \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \right\} \subseteq g = sl(2). \]

**Lemma.** \( f \in P(g) \mapsto f|_h \) is an injection. If \( f \in P(g)^G \), then \( f|_h \) is symmetric.

**Proof.** A polynomial function is determined by its values on an open set. Use the open set given by the diagonalizable matrices. Any matrix in this set can be conjugated to one in \( h \). An invariant polynomial is therefore determined by its values on \( h \). The fact that the image of the restriction map consists of symmetric polynomials follows from the fact that the subgroup \( W \subseteq G \) spanned by \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) leaves \( h \) invariant and acts by
\[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix} \]
□
Proposition. The restriction map
\[ \text{Res}: \mathcal{P}(\mathfrak{g})^G \rightarrow \mathcal{P}(\mathfrak{h})^W \]
is an isomorphism.

Proof. The previous lemma shows that the map is injective and the image is contained in the space of symmetric polynomials. This is a polynomial algebra in the generator \(p(a) = a^2\). We need to see that this comes from an element in \(\mathcal{P}(\mathfrak{g})^G\). Note that
\[ P(X) := -\det(X) = -\det \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a^2 + bc \]
is \(SL(2)\) invariant and restricts to \(p\).

\[ \square \]

1.13. Highest weight modules for \(SL(2)\). We use the notation in example (b) of section 1.4. Let \(\lambda \in \mathbb{C}\), and define a representation \(C_\lambda = \mathbb{C}v_\lambda\) of \(b\) by
\[ \pi(e)v_\lambda = 0, \quad \pi(h)v_\lambda = \Lambda v_\lambda. \]
Then we can form the module \(M(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\Lambda\). Then \(\mathcal{U}(\mathfrak{g})\) acts by multiplication on the left. It is a highest weight module for \(\mathfrak{g}\). This means that it is generated by an eigenvector for \(h\) which is annihilated by \(e\). It is also universal, in the sense that if any module \((\pi,V)\) of \(\mathfrak{g}\) has a vector \(v\) which transforms like \(C_\Lambda\), then there is a map
\[ \phi: M(\Lambda) \rightarrow V, \quad \phi(v_\lambda) = v, \phi(X \cdot w) = X \cdot \phi(w) \]
This is clear, the map is \(\phi(x \otimes v_\lambda) := \pi(x)v\).

We analyze the module \(M(\Lambda)\). The PBW theorem implies that a basis is given by
\[ f^n \otimes v_\Lambda. \]
The following relations hold in the universal enveloping algebra.
\[ hf^n = f^n(h - 2n), \quad ef^n = f^n e + n f^{n-1}(h - n + 1). \]
Thus in \(M(\Lambda)\),
\[ h \cdot f^n \otimes v_\Lambda = (\Lambda - 2n) f^n \otimes v_\Lambda, \quad e \cdot f^n \otimes v_\Lambda = n(\Lambda - n + 1) f^n \otimes v_\Lambda. \]
We would like to know when \(M(\Lambda)\) is reducible. Let \((0) \subset W \subset V\) be a nontrivial submodule. Any vector \(w \in W\) has a decomposition
\[ w = \sum w_i, \quad h \cdot w_i = \lambda_i w_i. \]
We claim that \(w_i \in W\) as well. Indeed suppose not. Let \(w := w_1 + \cdots + w_n \in W\), so that none of the \(w_i \in W\) have distinct eigenvalues and the \(n\) is minimal with these properties. Then
\[ h \cdot \sum w_i = \sum \lambda_i w_i \in W. \]
Assume as we may that \(\lambda_1 \neq 0\). Then \(h \cdot w - \lambda_1 w \in W\) has the same properties as \(w\) but has fewer terms, a contradiction. Next we claim \(W\) must have a highest weight different from \(v_\Lambda\). Indeed, \(e\) raises the eigenvalue of \(h\) by 2,
all the eigenvalues of $h$ on $M(\Lambda)$ are of the form $\Lambda - 2n$ with $n \geq 0$, so if $e$ has no kernel, it must eventually contain $v_{\Lambda}$. But then $W = M(\Lambda)$. Thus $W$ contains a $v_{\Lambda - 2n}$ which is annihilated by $e$. It follows that $\Lambda = n - 1$. In this case we get an exact sequence

$$0 \rightarrow M(-n - 2) \rightarrow M(n) \rightarrow F(n) \rightarrow 0,$$

and $M(-n - 2)$ is irreducible.

**Exercise 6.** Show that the exact sequence in (1.13.5) does not split.