1. Introduction

1.1. Lie algebras.

Definition. A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with a Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is

1. bilinear, i.e. $[\lambda x, y] = \lambda [x, y] = [x, \lambda y]$
2. skew symmetric $[x, y] = -[y, x]$
3. satisfies the Jacobi identity

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0.$$

1.2. Examples.

1. Let $A$ be any associative algebra. Then define

$$[a, b] := ab - ba.$$

The main such example is

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K}) \quad (= M_n(\mathbb{K}))$$

$n \times n$ matrices with matrix multiplication. More generally, for any vector space $V$, we can define $\mathfrak{gl}(V) := \{ A : V \to V \text{ linear} \}$ with the same bracket as before.

2. Let

$$\mathfrak{sl}(2, \mathbb{K}) := \{ X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{gl}(2, \mathbb{K}) : \text{tr}X = 0 \}$$

with the Lie algebra structure inherited from $\mathfrak{gl}(2)$. Then $\mathfrak{sl}(2)$ is a Lie algebra. We also say that it is a subalgebra of $\mathfrak{gl}(2)$.

3. Let $p_1, \ldots, p_n, q_1, \ldots, q_n, z$ be a basis of $\mathbb{K}^{2n+1}$. The Heisenberg algebra is defined by the bracket relations

$$[p_i, q_j] = \delta_{i,j}z, \quad [p_i, p_j] = [q_i, q_j] = [p_i, z] = [q_j, z] = 0.$$

Some of you remember the operators with relations

$$\{p_1, \ldots, p_n, q_1, \ldots, q_n\}, \quad [p_i, p_j] = 0, \quad [q_i, q_j] = 0, \quad [p_i, q_i] = h\delta_{ij}.$$
from mathematical physics. The \( p \)'s represent positions and \( q \)'s momentums. A representation is
\[
\begin{bmatrix}
0 & p_1 & \cdots & p_n & z \\
0 & 0 & 0 & q_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & q_1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(4) Let \( U \subseteq \mathbb{R}^n \) be an open set. A vector field is a linear map
\[ X : C^\infty(U) \to C^\infty(U) \]
satisfying
\[ X(f_1 \cdot f_2) = X(f_1) \cdot f_2 + f_1 \cdot X(f_2). \]

**Exercise 1.** Show that there are \( a_i \in C^\infty(U) \) such that
\[ X(f) = \sum a_i \partial_i f. \]

The set of vector fields \( \text{Vec}(U) \) is a vector space over \( C^\infty(U) \) under pointwise addition and multiplication. It is also a Lie algebra under
\[ [X_1, X_2] := X_1 \circ X_2 - X_2 \circ X_1. \]

The algebra generated by \( \text{Vec}(U) \) is called the algebra of differential operators.

In dynamical systems/control theory, one often considers the Lie subalgebras generated by a set of vector fields. They are usually infinite dimensional.

### 1.3. Structure constants

The previous example suggests that we can describe a Lie algebra by choosing a basis \( \{e_\alpha\} \). The Lie bracket is determined by
\[ [e_\alpha, e_\beta] = \sum c^k_{\alpha\beta} e_k. \]
The \( c^k_{\alpha\beta} \) are called the structure constants. They have to satisfy \( c^k_{\alpha\beta} = -c^k_{\beta\alpha} \) and relations coming from the Jacobi identity.

**Exercise 2.** Translate the Jacobi identity into a relation among the \( c^k_{ij} \).

**Definition.** Let \( g \) and \( h \) be two Lie algebras. A Lie (algebra) homomorphism is a linear map \( \phi : g \to h \) which commutes with the brackets, i.e.
\[ \phi([x, y]) = [\phi(x), \phi(y)]. \]
It is called an isomorphism if \( \phi^{-1} \) (exists and) is also a Lie algebra homomorphism.

If \( \phi \) has an inverse as a linear map, then the inverse is automatically a Lie homomorphism. To classify Lie algebras means to list them up to isomorphism.
Exercise 3. Assume $\mathbb{K} = \mathbb{C}$ (or algebraically closed of characteristic zero). Classify all Lie algebras of dimension less than or equal to 3.

Besides the abelian ones, in dimension 2 there is an algebra generated by $\{h, e\}$ with structure constants $[h, e] = e$. You should also find the Heisenberg algebra $H_3$

$$\text{span} \{x, y, z\}, \quad [x, y] = z, \quad [x, z] = [y, z] = 0.$$  

A realization is given by $3 \times 3$ matrices

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \quad x, y, z \in \mathbb{C}$$

with the usual bracket.

Another interesting algebra is $sl(2, \mathbb{C})$, given by

$$\text{span} \{e, h, f\}, \quad [h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \quad \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

with the usual bracket

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

1.4. Generators and relations. Another way to describe Lie algebras is by generators and relations. Recall that the free algebra (with unit) generated by a set $X$ is a pair $(\mathcal{F}, i)$ consisting of an algebra $\mathcal{F}$ and a map $i : X \to \mathcal{F}$ such that if $\theta : X \to \mathcal{G}$ is any map to an associative algebra with unit $\mathcal{G}$, then there is a unique algebra homomorphism $\Theta : \mathcal{F} \to \mathcal{G}$ such that $\theta = \Theta \circ i$, and $\Theta(1) = 1$. It is straightforward to show that the free algebra on $X$ exists and is unique up to isomorphism. For the existence, one takes $V$, a vector space with basis indexed by $X$. Let $T^i(V) = V \otimes \cdots \otimes V$.

Then

$$T(V) := \mathbb{K} \oplus V \oplus T^2(V) \oplus \ldots$$

with the obvious $i$ and multiplication

$$(1.4.1) \quad (x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots y_m) = x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m.$$  

is the desired free algebra. The algebra with generators $X$ and relations $\mathcal{R}$ (a subset of elements of $\mathcal{F}_X$) is (by definition) the quotient of $\mathcal{F}_X$ by the two sided ideal generated by $\mathcal{R}$. For example the symmetric algebra $S(V)$ is the algebra with generators $X$ and relations $x_1 \otimes x_2 - x_2 \otimes x_1 = 0$.

For a Lie algebra, the analogous definitions apply. But it is a bit more awkward to show existence.

Exercise 4. Let $\mathcal{L}_X$ be the Lie subalgebra generated by $i(X) \subset \mathcal{F}_X$. Show that $(\mathcal{L}, i)$ is a free algebra.
Here is an important set of examples of Lie algebras given by generators and relations. Let $A$ be an $n \times n$ matrix of rank $\ell$. It is called a generalized Cartan matrix if it also satisfies

1. $a_{ii} = 2$,
2. $a_{ij}$ are non-positive integers,
3. if $a_{ij} = 0$ then $a_{ji} = 0$.

Let $h$ be a vector space of dimension $2n - \ell$, and $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset h^*$, be a set of independent functionals, and $\Pi^V = \{\alpha_1^V, \ldots, \alpha_n^V\} \subset h$ be linearly independent sets in duality with respect to the standard pairing $\langle \cdot, \cdot \rangle : h \times h^* \to \mathbb{C}$ satisfying $\langle \alpha_i^V, \alpha_j \rangle = \delta_{ij}$. Define the Lie algebra $\tilde{g}(A)$ to be the Lie algebra with generators $\{e_i, h, f_j\}$ subject to the relations

$$[e_i, f_j] = \delta_{ij} \alpha_i^V, \quad [h, h'] = 0, \quad [h, e_j] = \langle h, \alpha_i \rangle e_j, \quad [h, f_j] = -\langle h, \alpha_i \rangle f_j.$$ 

Define $g(A) := \tilde{g}(A)/\mathfrak{r}$, where $\mathfrak{r}$ is the largest ideal so that $\mathfrak{r} \cap h = (0)$.

Exercise 5. Show that $\mathfrak{r}$ exists.

Examples. The matrix $A = (2)$ gives $g(A) \cong sl(2)$. The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ gives rise to $g(A) \cong sl(3)$. But $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ gives rise to an infinite dimensional algebra called the affine Kac-Moody algebra $\tilde{A}_1$. The previous examples have been known and studied for at least 100 years. The Kac-Moody algebras have been studied very intensely only in the last 40 years and they have important applications in number theory and mathematical physics. Quantum groups are variants of the above ideas.

Exercise 6. Verify the assertion for $A = (2)$.

1.5. Relations to Lie groups. One of the main uses of Lie algebras is through their representations. If $G$ is a group we say $(\pi, V)$ is a representation if $V$ is a vector space over $\mathbb{K}$ and $\pi$ is a group homomorphism

$$\pi : G \to Aut(V).$$

If $g$ is a Lie algebra, $(\pi, V)$ is a representation if

$$\pi : g \to End(V)$$

is linear and $\pi([x, y]) = [\pi(x), \pi(y)]$.

Usually the representations we consider in this course are finite dimensional.

A representation is called irreducible if there is no $0 \not\subseteq W \not\subseteq V$ which is invariant under $\pi(g)$. It is called completely reducible if for any $\pi(g)$-invariant $W \subseteq V$, there is a $\pi(g)$-invariant $W'$ such that $W \cap W' = \{0\}$ and $g = W + W'$. 
Examples.

(1) \( g = \mathbb{C}, \pi(1) = \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} \) is completely reducible.

(2) \( g = \mathbb{C}, \pi(\text{one}) = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \) is not.

A representation typically will have a Jordan-Hölder type composition series

\[ 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \]

such that

(a) \( V_i \) is invariant

(b) \( V_{i+1}/V_i \) is irreducible.

There may be more than one such decomposition.

There is a strong relation between Lie groups and their Lie algebras. Let \( GL(n, \mathbb{R}) := \{ g \in gl(n, \mathbb{R}) \mid \det(g) \neq 0 \} \) and \( G \subseteq GL(n, \mathbb{R}) \) be a closed subgroup. Recall

\[ \exp : gl(n, \mathbb{R}) \to GL(n, \mathbb{R}) \]

\[ \exp X = e^X := \sum_{n=0}^{\infty} \frac{1}{n!} X^n \]

Define

\[ g := \left\{ X \in gl(n, \mathbb{R}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R} \right\}. \]

**Proposition.** \( g \) is a Lie algebra.

A proof can be found in Godement’s notes on Lie groups and algebras, [Gd]. Here is a sketch. We need to show that if \( X, Y \in g \), then \( \lambda X, X + Y, [X,Y] \in g \) as well. The first one is clear. For the second one, we compute \( e^{tX} \cdot e^{tY} : \)

\[
(1 + \frac{t}{1} X + \frac{t^2}{2!} X^2 + t^3 \ldots)(1 + \frac{t}{1} Y + \frac{t^2}{2!} Y^2 + t^3 \ldots) =
1 + t(X + Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + t^3 \ldots
\]

With a little work we can rewrite this relation as

\[ e^{tX} e^{tY} = e^{t(X+Y)+O(t^2)}. \]

(See Helgason for details in the case of an arbitrary Lie group). Then

\[ [e^{\frac{t}{n} X}, e^{\frac{t}{n} Y}] = e^{t(X+Y) + \frac{1}{n} O(t^2)} \]

As \( n \to \infty \), the RHS tends to \( e^{t(X+Y)} \) while the LHS is always in \( G \). Because \( G \) is assumed closed, the claim about \( X + Y \) follows. For the claim about \( [X,Y] \), one computes \( e^{tX} e^{tY} e^{-tX} e^{-tY} \) and applies a similar trick. The relevant relation is

\[ \left( e^{\frac{t}{n} X} e^{\frac{t}{n} Y} e^{-\frac{t}{n} X} e^{-\frac{t}{n} Y} \right)^n = e^{2[X,Y]+O(\frac{1}{n})}. \]
So to each closed subgroup one can attach a Lie algebra. Building on these ideas one can show that closed subgroups of $GL(n, \mathbb{R})$ are Lie groups, i.e. groups that have a differentiable structure so that multiplication and inverse map are differentiable. There is a close connection between Lie groups and their Lie algebras. For example, in order to study representations of the Lie group, one can study the corresponding Lie algebra representations. These are much easier to deal with because you are doing linear algebra.

**Summary of properties of $exp$ and $log$**

$$e^X = \sum \frac{X^n}{n!}.$$  

(1) if $X$ is a matrix, its norm is $||X|| = (\sum |x_{ij}|^2)^{1/2}$ or $\max |x_{ij}|$. The main point is that we need $||X + Y|| \leq ||X|| + ||Y||$, and $||XY|| \leq ||X|| \cdot ||Y||$.

(2) $e^0 = I$.

(3) $(X)^* = e^{X^*}$.

(4) $e^{X+Y} = e^X \cdot e^Y$ if $[X,Y] = 0$.

(5) $Ce^X e^{-1} = e^{CX} e^{-1}$.

(6) $||e^X|| \leq e^{||X||}$.

$$\log A = \sum (-1)^{m+1} \frac{A^m}{m} \quad \text{for } ||A - I|| < 1$$

(1) $||\log(I + B) - B|| \leq c||B||^2$ for some $c > 0$ and any $||B|| < 1/2$.

Some consequences are that all closed subgroups of $GL(n, \mathbb{R})$ have a $C^\infty$ structure, in fact are analytic (Lie Groups). They are studied via their algebras. These are called linear groups.

**Note:** Not all Lie groups are of this form. The *metaplectic group* $Mp(2, \mathbb{R})$ is

$$\{(g, \epsilon) : g \in SL(2, \mathbb{R}), \epsilon \text{ analytic function on the upper half plane, } cz + d = e^2\}$$

The multiplication rule is $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1(g_2 \cdot z)\epsilon_1(z))$. Here

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad g \cdot z = \frac{az + b}{cz + d}.$$  

The fact that this is a group follows from the property of $j(g, z) := cz + d$ satisfying $j(g_1g_2, z) = j(g_1, g_2 \cdot z)j(g_2, z)$.

**Example.** Recall that $SO(3)$ is the group that preserves the form

$$\langle v, w \rangle := \sum_{1 \leq i \leq 3} v_i w_i,$$

in the sense that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle.$$
It is a closed subgroup so the previous proposition applies.

\[(1.5.4) \quad g := \{ X : e^{tX} \in G \text{ for all } t \}\]

\[(1.5.5) \quad \langle e^{tX} \cdot v, e^{tX} w \rangle = \langle v, w \rangle \quad \forall t \in \mathbb{R}.\]

Differentiate in \(t\):

\[\langle Xv, w \rangle + \langle v, Xw \rangle = 0\]

so \(X\) is skew hermitian. The converse is also true, \textit{i.e.} if \(X\) is skew symmetric, then \(e^X\) is orthogonal.

**Example.** Let \(U(n) \subset GL(n, \mathbb{C})\) be the unitary group, the subgroup of matrices that preserve the sesquilinear form

\[\langle v, w \rangle := \sum_{1 \leq i \leq n} v_i \overline{w_i},\]

\(\text{i.e.}\)

\[U(n) = \{ g \in GL(n, \mathbb{C}) : \langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle \}.\]

Its Lie algebra is the Lie algebra of skew hermitian matrices,

\[u(2) := \{ X \in gl(n, \mathbb{C}) : X + X^* = 0 \}.\]

1.6. **Invariants of group actions.** Suppose \((\pi, V)\) is a representation of a topological group on a finite dimensional space. We would like to describe the orbits \(\pi(G) \cdot v\). One way to do this is to observe that \(G\) acts on \(R := \mathcal{P}[V]\), polynomial functions on \(V\), by

\[g \cdot f(v) := f(g^{-1}v).\]

Suppose \(f \in R\) is such that \(g \cdot f = f\) for all \(g \in G\). Denote by \(R^G\) the space of all such polynomials. Such \(f\) are constant on the orbits. We could hope that the “levels” of \(R^G\) actually completely describe the orbits.

**Example.** Suppose \(G = GL(n)\) is acting on \(V = gl(n)\) by

\[\pi(g)X := gXg^{-1}.\]

Describing the orbits is the same as classifying matrices up to similarity. we know that this is given by Jordan canonical forms. The ring of invariants \(R^G\) is generated by the polynomials \(p_i(X)\) which occur in the expansion

\[\det(tI - X) = t^n - p_{n-1}(X)t^{n-1} + \cdots + (-1)^np_0(X).\]

Recall that \(p_{n-1}(X) = \text{tr} X\) and \(p_0(X) = \det(X)\). The “levels” of these polynomials classify semisimple matrices.
Example. Let $V_n$ be the space of homogeneous polynomials in two variables $x, y$. These are called $n$-ary forms. Then $SL(2)$ acts on $V_n$ by

$$\pi_n\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)x^r y^s = (ax + cy)^r (bx + dy)^s.$$ 

One of the main problems of classical invariant theory is to describe $R^G$. For $n = 2$ and $3$ this is well known, but beyond that it is unknown. One of the reasons noetherian rings were invented is to prove that $R^G$ is finitely generated.

We show that the representation $V_n$ is an irreducible representation of $SL(2)$ and that $(\pi_n, V_n)$ are all irreducible representations up to equivalence. We compute the representation of $sl(2)$ attached to $(\pi_n, V_n)$. If $X \in g$, define

$$(1.6.1) \quad \pi(X)v := \frac{d}{dt}|_{t=0} \pi_n(e^{tX})v.$$ 

We get

$$(1.6.2) \quad \pi_n(e)x^r y^s = sx^{r+1}y^{s-1}, \quad \pi_n(h)x^r y^s = (r-s)x^r y^s, \quad \pi_n(f)x^r y^s = rx^{r-1}y^{s+1}.$$ 

A change of basis (actually just a rescaling) shows that this representation is equivalent to the one in section 1.11. Suppose that $W \subset V$ is a $G$-invariant subspace. Then $W$ is also $g$ invariant, so either $0$ or $V_n$. Thus $V_n$ is irreducible as a $G$ module. On the other hand, if $V$ is a finite dimensional irreducible representation of $G$, then a general theorem on Lie groups implies that the representation is $C^\infty$ so it differentiates to a representation of $g$. If it is not irreducible, then let $0 \subset W \subset V$ be a $g$ invariant subspace. Because the exponential map is onto, $W$ is a $G$ invariant subspace so either equal to $0$ or $V$. But then section 1.11 shows that $V = V_n$.

1.7. Spherical harmonics. Let $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. This acts on $C^\infty$ functions on $\mathbb{R}^3$ by differentiation. One problem that shows up a lot in analysis is to solve

$$\Delta f = \lambda f.$$ 

It is known that such solutions have to be analytic (called regularity theorem). Let $C^\omega$ be the vector space of analytic functions. Then the group $SO(3)$ acts

$$(1.7.1) \quad \pi : G \to Aut(C^\omega)$$

$$(1.7.2) \quad (\pi(g)f)(x) := f(g^{-1} \cdot x).$$ 

You can check easily that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$, $\pi(1) = I_d$ and so on. Furthermore,

$$\Delta(\pi(g)f)(x) = \Delta(f)(g^{-1}x).$$ 

Thus the space $C^\omega_\chi := \{ f \in C^\omega \mid \Delta f = \lambda f \}$ is invariant under all $\pi(g)$, so it is a representation.

Problem: Describe the representation of $G$ on $C^\omega_\chi$. 
Exercise 7. If $X$ is skew hermitian, then $e^X$ is orthogonal.

We can then define a representation of $\mathfrak{g}$ on $C_X^\omega$:

$$\pi(X) f(x) := \frac{d}{dt} \bigg|_{t=0} f(e^{-tx} \cdot x).$$

Now $\mathfrak{g}$ is generated by three elements

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. $$

satisfying

$$[e_1, e_2] = -e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = -e_2.$$

Their exponentials are

\begin{align*}
(1.7.3) \quad e^{te_1} &= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
(1.7.4) \quad e^{te_2} &= \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \\
(1.7.5) \quad e^{te_3} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.
\end{align*}

so

\begin{align*}
(1.7.6) \quad e_1 \cdot f(x) &= -x_2 \partial_1 + x_1 \partial_2 = (x_1 \partial_2 - x_2 \partial_1)(f) \\
(1.7.7) \quad e_2 \cdot f(x) &= (x_1 \partial_3 - x_3 \partial_1)(f) \\
(1.7.8) \quad e_3 \cdot f(x) &= (x_2 \partial_3 - x_3 \partial_2)(f).
\end{align*}

Definition. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$, and let $\mathbb{K} \subseteq \mathbb{L}$ be a field extension. Then we can define the extension

$$\mathfrak{g}_L := \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{L}.$$ 

If $\mathbb{K} = \mathbb{R} \subseteq \mathbb{L} = \mathbb{C}$ we write $\mathfrak{g}_c$ and call it the complexification of $\mathfrak{g}$.

If $\{e_\alpha\}$ is a basis for $\mathfrak{g}$, then $\mathfrak{g}_L$ has a basis $\{e_{\alpha}\}$ with the same

$$[e_\alpha, e_\beta] = \sum \epsilon_{\alpha\beta}^i e_i$$

but elements $x = \sum m_\alpha e_\alpha$ with $m_\alpha \in \ell$ instead of $k$.

Exercise 8. Show that so(3)$_C \simeq sl(2, \mathbb{C})$. $\sqrt{-1} e_1 \leftrightarrow h$. 

1.8. We return to the example. We want to describe the action of $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ on $C^\infty(\mathbb{R}^3)$, analytic functions with complex values. We will analyze

$$\mathcal{P} := \mathbb{C}[x_1, x_2, x_3],$$

polynomials with complex coefficients.

We know that $\Delta$ commutes with $x_i \partial_j - x_j \partial_i$. Let $Q(x) = x_1^2 + x_2^2 + x_3^2$. Then

$$(x_i \partial_j - x_j \partial_i)[Q \cdot f] = Q \cdot (x_i \partial_j f - x_j \partial_i f).$$

We can express this as

$$[x_i \partial_j - x_j \partial_i, m_Q] = 0$$

where $m_Q$ is the operator given by multiplication by $Q$. Since $so(3)$ commutes with $\Delta$ and $m_Q$, it also commutes with all the brackets of these elements. We find

$$[\Delta, m_Q] = 4\left( x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + \frac{3}{2} \right).$$

We find that

$$\frac{1}{2} \Delta \leftrightarrow f, \quad -\frac{1}{2} m_Q \leftrightarrow e, \quad E \leftrightarrow \frac{3}{2} \leftrightarrow h$$

forms an $sl(2, \mathbb{C})$.

1.9. The six elements form a Lie algebra,

$$so(3) \times sl(2) \subseteq \text{End}(\mathcal{P})$$

or $sl(2) \times sl(2)$ if we complexify $so(3)$:

(1.9.1) \quad h \leftrightarrow -2i(x_1 \partial_2 - x_2 \partial_1) = -2ie_1

(1.9.2) \quad e \leftrightarrow e_2 + ie_3

(1.9.3) \quad f \leftrightarrow e_2 - ie_3.

1.10. Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras. We can form a new algebra

$$\mathfrak{g}_1 \times \mathfrak{g}_2 = \{(x_1, x_2) : x_i \in \mathfrak{g}_i\}$$

with operations

(1.10.1) \quad \alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2)

(1.10.2) \quad [(x_1, y_1), (x_2, y_2)] = [(x_1, x_2), [y_1, y_2]].

**Question:** What do representations of $\mathfrak{g}_1 \times \mathfrak{g}_2$ look like?

For example if $(\pi_i, V_i)$ are representations of $\mathfrak{g}_i$, then $(\pi_1 \times \pi_2, V_1 \boxtimes V_2),

$$(\pi_1 \otimes \pi_2)(x, y) = \pi_1(x) \otimes I + I \otimes \pi_2(y)$$

defines a new representation. A general theorem states that irreducible representations of a product are exterior tensor products of irreducible representations of the individual factors.
1.11. Irreducible representations of \(\mathfrak{sl}(2)\). Say we are talking about finite dimensional representations \(\pi, F\). Then \(\pi(h)\) has an eigenvector \(v = v_\lambda \neq 0\)

\[
\pi(h)v = \lambda v.
\]

Then \(\pi(e)v\) satisfies

\[
\pi(h)\pi(e)v = \pi(e)\pi(h)v + \pi([h,e])v = (\lambda + 2)\pi(e)v.
\]

So since \(\dim F < \infty\), there is \(v_0 \neq 0\) such that

\[
\pi(h)v_0 = \Lambda v_0 \quad \pi(e)v_0 = 0.
\]

Let \(W := \text{span}\{\pi(f)^kv_0 := v_k\}\).

**Claim:** \(W\) is \(g\)-invariant.

**Proof.** The following relations hold:

(1.11.1)

\[
\pi(h)v_k = \pi(h)\pi(f)^kv_0 = \\
= \pi(f)\pi(h)\pi(f)^{k-1}v_0 - 2\pi(f)^kv_0 = \cdots = (\Lambda - 2k)\pi(f)^kv_0,
\]

So \(v_k\) is a \((\Lambda - 2k)\) eigenvector for \(\pi(h)\). Then

(1.11.2)

\[
\pi(e)v_k = \pi(e)\pi(f)^kv_0 = \pi(f)\pi(e)\pi(f)^{k-1}v_0 + \pi(h)\pi(f)^{k-1}v_0 \\
= \pi(f)\pi(e)\pi(f)^{k-1}v_0 + (\Lambda - 2k + 2)v_{k-1}
\]

(1.11.3)

\[
= \cdots (\Lambda + \Lambda - 2 + \cdots + \Lambda - 2k + 2)v_{k-1} = \left(k\Lambda - 2 \cdot \frac{k(k-1)}{2}\right)v_{k-1}
\]

(1.11.4)

\[
= k(\Lambda - k + 1)v_{k-1}.
\]

Since \(W \neq 0\) it follows that \(W = V\). Next recall that we want \(V\) to be finite dimensional. Let \(N\) be such that \(v_{-N} \neq 0\) but \(\pi(f)v_{-N} = 0\). Then

(1.11.5)

\[
(\Lambda - 2N)v_{-N} = \pi(h)v_{-N} = -\pi(f)\pi(e)v_{-N} = -N(\Lambda - N + 1)v_{-N}
\]

So we get the equation \((N + 1)\Lambda = N(N + 1)\) and therefore \(\Lambda = N\).

The conclusion is that for each integer \(N \geq 0\) there is an irreducible module

\[
F_N = \{v_{-N}, v_{-N+2}, \ldots, v_N\}.
\]

In fact we get more. For each \(\nu \in \mathbb{C}\) we can construct a module

\[
V_\nu = \{v_{\nu - 2n}\}
\]
with action
(1.11.7) \( \pi(h)v_{\nu-2n} := (\nu - 2n)v_{\nu-2n} \)
(1.11.8) \( \pi(f)v_{\nu-2n} := v_{\nu-2n-2} \)
(1.11.9) \( \pi(e)v_{\nu-2n} := n(\nu - n + 1)v_{\nu-2n+2} \).

If \( \nu = N \in \mathbb{N} \), then this module has an irreducible quotient \( F_N \) and a submodule

\[
0 \to V_{-N-2} \to V_N \to F_N \to 0.
\]

The modules \( V_\nu \) are “characterized” by the existence of a vector \( v_\nu \) satisfying

\[
\pi(h)v_\nu = \nu v_\nu, \quad \pi(e)v_\nu = 0.
\]

This is called a highest weight vector. \( \square \)

**Remark:** General irreducible representations of \( sl(2, \mathbb{C}) \) are much more complicated.

**1.12.** Back to the problem of decomposing \( \mathcal{P} \). We look for eigenvectors of

\[
-2i(x_1\partial_2 - x_2\partial_1), \left(x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + \frac{3}{2}\right)
\]

annihilated by

\[
\Delta \text{ and } x_1\partial_3 - x_3\partial_1 + i(x_2\partial_3 - x_3\partial_2).
\]

**Note:** \( \Delta \) acts locally nilpotently on \( \mathcal{P} \). This means that for any element \( p \in \mathcal{P} \), there is \( n \) (depending on \( p \)) such that \( \Delta^n p = 0 \). The second operator in (a) has eigenvalue \( r + \frac{3}{2} \) on \( \mathbb{C}_r[x_1, x_2, x_3] \), homogeneous polynomials of degree \( r \). The first operator in (a) acts by \( 2(-\beta + \alpha) \) on

\[
(x_1 - ix_2)^\alpha(x_1 + ix_2)^\beta x_3^\gamma.
\]

It is better to use the basis

\[
(x_1 - ix_2)^\alpha(x_1^2 + x_2^2 + x_3^2)^m (x_1 - ix_2)^\alpha x_3(x_1^2 + x_2^2 + x_3^2)^m x_3.
\]

We find that the second operator in (b) annihilates

\[
(x_1 + ix_2)^\alpha(x_1^2 + x_2^2 + x_3^2)^m
\]

and \( \Delta \) annihilates these only when \( m = 0 \).

**Exercise 9.** Show the following.

1. \( \mathcal{P} \) decomposes

\[
\oplus F_\alpha \otimes V_{\alpha + \frac{3}{2}}
\]

2. The spaces \( \mathcal{H}_\alpha \) generated by \( (x_1 + ix_2)^\alpha \) are the only solutions to

\[
\Delta f = \Lambda f \text{ where } \Lambda = 0.
\]

\( \square \)

**1.13.** Some basic constructions of representations.

**1.13.1. Direct sums.** Let \( (\pi, V) \), \( (\rho, W) \) be representations of \( \mathfrak{g} \). Then \( (\pi \oplus \rho, V \oplus W) \) given by \( (\pi \oplus \rho)(x) = (\pi(x), \rho(x)) \).
1.13.2. **Exterior product.** \( g_1, g_2 \) Lie algebras. We can form \( g_1 \times g_2 \). If \( (\pi_i, V_i) \) are representations, \( (\pi_1 \boxtimes \pi_2, V_1 \boxtimes V_2) \) is \( (\pi_1 \boxtimes \pi_2)(x_1, x_2) := \pi_1(x_1) \otimes 1 + 1 \otimes \pi_2(x_2) \). If \( g_1 \simeq g_2 \simeq g \), then \( g \hookrightarrow g_1 \times g_2 \) as the diagonal. The restriction of \( \pi_1 \boxtimes \pi_2 \) to \( g \) is called \( \pi_1 \otimes \pi_2 \)

\[
(\pi_1 \otimes \pi_2)(x) := \pi_1(x) \otimes 1 + 1 \otimes \pi_2(x)
\]
and is sometimes referred to as *interior tensor product.*

1.13.3. **Adjoint representation.**

(1.13.1) \quad ad : g \to End(g)

(1.13.2) \quad adx(y) := [x, y].

The Jacobi identity plays two roles:

(a) \( ad \) is a Lie homomorphism, \( ad[x, y] = [adx, ady] \).

(b) the image of \( ad \) is in \( Der(g) : adx([y, z]) = [adx(y), z] + [y, adx(z)] \).