Semisimple Lie Algebras
Math 649, 2013
Root Systems

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REFERENCES:  Bourbaki chapter 6, Humphreys chapter III
Simple Roots

**Definition 1**

A subset $\Pi \subset R$ is called a base if

(I) $\Pi$ is a basis of $V$

(II) Any $\beta \in R$ can be written $\beta = \sum n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}$, either all positive or all negative. The roots in $\Pi$ are called simple.

**Properties:**

1. A root is called positive if all the $n_\alpha \geq 0$, negative if all the $n_\alpha \leq 0$. Then $R = R^+ \cup R^-$, $R^+ \cap R^- = \emptyset$, where $R^\pm$ are the positive (negative) roots.
2. If $\alpha, \beta \in R^+$, then $\alpha + \beta \in R^+$, or it is not a root.
3. We say $\alpha \leq \beta$ if $\beta - \alpha \in R^+$.
4. If $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root.
Proof.

If \( \gamma = \alpha - \beta \) is a root, either \( \gamma \in R^+ \), or \( -\gamma \in R^+ \). Either way, it violates (II).
Theorem 2

Every root system has a base.

Proof.

Let $H_\alpha$ be the hyperplane where $\tilde{\alpha}$ is zero. Let $0 \neq h_0 \in V \setminus \bigcup H_\alpha$. Then $\tilde{\alpha}(h_0) \neq 0$ for all $\alpha \in R$. (1)

Let $R^+ := \{\alpha \mid \tilde{\alpha}(h_0) > 0\}$. Then $R = R^+ \bigsqcup R^-$ where $R^- = -R^+$. We say $\alpha \in R^+$ is indecomposable if $\alpha$ cannot be written as $\alpha = \beta + \gamma$, $\beta, \gamma \in R^+$. (2)
Claim: The set of indecomposable elements is a base.

Proof.

Call this set \( \Pi \).

(1) Each \( \alpha \in R^+ \) is an \( \mathbb{N} \)-linear combination of elements in \( \Pi \).

(2) \( \alpha, \beta \in \Pi \) distinct, then \( \alpha - \beta \) is not a root and \( (\alpha, \beta) \leq 0 \). Otherwise \( \beta = \alpha + \gamma \) or \( \alpha = \beta + \gamma \) with \( \gamma \in R^+ \).

(3) \( \Pi \) forms a linear independent set. Suppose \( \sum_{\alpha \in \Pi} n_\alpha \alpha = 0 \).

Then we get a relation \( r = \sum_{\alpha \in \Pi} n_\alpha \alpha = \sum m_\beta \beta, \ n_\alpha, m_\beta > 0 \), and the two sets are disjoint. But then

\[
(r, r) = \sum n_\alpha m_\beta (\alpha, \beta) \leq 0. \quad \text{So } r = 0. \quad \text{But then}
\]

\[
0 = (r, h_0) = \sum n_\alpha \langle \alpha, h_0 \rangle = \sum m_\beta \langle \beta, h_0 \rangle, \quad \text{so all}
\]

\( n_\alpha, m_\beta = 0 \).
Proposition 1

Each base is obtained in this fashion.

Proof.

Choose $h_0$ such that $\langle \alpha, h_0 \rangle > 0$ for all $\alpha \in \Pi$. Let $R^\pm$ be the positive and negative systems corresponding to $h_0$. Clearly

$$R^+ = \left\{ \beta \in R \mid \beta = \sum_{\alpha \in \Pi} n_\alpha \alpha, \; n_\alpha \in \mathbb{N} \right\}.$$  \hspace{1cm} (3)
A Weyl chamber is a connected components $C$ of $V - \bigcup_{\alpha \in R} H_\alpha$. Recall that a regular element $h_0 \subset V$ is an element so that

$$\langle \alpha, h_0 \rangle \neq 0 \quad \text{for any} \quad \alpha \in R. \quad (4)$$

Each $h_0$ regular determines a Weyl chamber $C(h_0)$

$$\{\text{Weyl chambers}\} \longleftrightarrow \{\text{Bases}\}. \quad (5)$$

**Lemma 3**

*The Weyl group $W(R)$ permutes the Weyl chambers.*

**Proof.**

Clear.
Lemma 4

If $\gamma \in R^+$ is not simple, there is $\alpha \in \Pi$ such that $\gamma - \alpha \in R^+$.

Proof.

There is $\alpha \in R^+$ such that $(\gamma, \alpha) > 0$. This implies the claim of the lemma. If $(\gamma, \alpha) \leq 0$ for all $\alpha \in \Pi$, then since

$$
\gamma = \sum_{\beta \in \Pi} n_\beta \beta, \quad n_\beta \geq 0,
$$

we get

$$(\gamma, \gamma) = \sum n_\beta (\gamma, \beta) \leq 0 \quad \text{so} \quad \gamma = 0.
$$
Corollary 5

Every root $\gamma \in R^+$ can be written as

$$\gamma = \alpha_1 + \cdots + \alpha_k \quad \alpha_j \in \Pi$$

such that $\alpha_1 + \cdots + \alpha_i$ is a root for each $i \leq k$.

Proof.

Exercise.
Lemma 6

Let $\alpha \in \Pi$. Then

$$s_\alpha(R^+) = (R^+ \setminus \{\alpha\}) \cup \{-\alpha\}. \quad (9)$$

Proof.

Clearly $s_\alpha(\alpha) = -\alpha$. Suppose $\gamma \neq \alpha$ is in $R^+$. Then

$$\gamma = \sum_{\beta \neq \alpha} n_\beta \cdot \beta + n_\alpha \cdot \alpha \quad (10)$$

$s_\alpha(\beta) = \beta + m \cdot \alpha$ with $m \geq 0$. There is at least one $n_\beta \neq 0$ in the first sum. It follows that the coefficient of $\beta$ in $s_\alpha(\gamma)$ is $> 0$. So all coefficients are $\geq 0$. Thus $s_\alpha(\gamma) \in R^+$. \qed
Corollary 7

If \( \rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha \), then

\[ s_\beta(\rho) = \rho - \beta \quad \text{for any} \quad \beta \in \Pi. \]  

(11)

Proof.

Exercise.
Theorem 8 (Main Theorem)

Let \( \Pi \) be a base of \( R \).

1. If \( h_0 \in V \) is regular, there exists \( w \in W \) such that
   \[
   \langle w(h_0), \alpha \rangle > 0 \quad \text{for all} \quad \alpha \in \Pi.
   \]

2. If \( \Pi' \subset R \) is another base, there is a \( w \in W \) such that
   \( w(\Pi') = \Pi \).

3. If \( \alpha \in R \) is any root, there is \( w \in W \) such that
   \[
   w\alpha \in \Pi. \tag{12}
   \]

4. \( W \) is generated by \( s_\alpha, \alpha \in \Pi \).

5. If \( w(\Pi) = \Pi \), then \( w = 1 \).
Proof.

Let $W'$ be the subgroup generated by $s_{\alpha}$, $\alpha \in \Pi$. We prove (1)-(3) for $W'$ and then (4).

(1) Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ where $R^+$ is defined by $\Pi$. Let $w \in W'$ be such that

$$\langle w(h_0), \rho \rangle \text{ is maximal.} \quad (13)$$

Let $\alpha \in \Pi$. Look at

$$\langle s_{\alpha} w(h_0), \rho \rangle = \langle w(h_0), \rho - \alpha \rangle \leq \langle w(h_0), \rho \rangle. \quad (14)$$

Thus $\langle w(h_0), \alpha \rangle \geq 0$ as claimed; in fact $> 0$ because $\langle w(h_0), \alpha \rangle \neq 0$ for any $\alpha$.

(2) Exercise.
Proof.

(3) Because of (2), it is enough to see that for any \( \alpha \in R \) there is at least one base it belongs to. Look at \( V \setminus H_\alpha \). Since \( R \) is reduced, all other \( H_\beta, \beta \neq \pm \alpha \) are distinct from \( H_\alpha \).

Choose \( h_0 \) so that

\[
(h_0, \alpha) = \varepsilon > 0 \quad |(h_0, \beta)| > \varepsilon \quad \beta \neq \pm \alpha.
\]

(15)

Then \( \alpha \in \Pi(h_0) \).

(4) Let \( \alpha \in R \). Then there is \( w \in W' \) such that

\[
w(\alpha) = \alpha_1 \in \Pi.
\]

We get

\[
s_\alpha = s_{w(\alpha_1)} = w \circ s_{\alpha_1} \circ w^{-1}
\]

(16)

so \( s_\alpha \) is a product of simple root reflections.

\[
(\ w = s_{\beta_1} \circ s_{\beta_2} \cdots \ )
\]
Proof.

(5) Suppose \( w(\Pi) = \Pi \). Write a minimal expression

\[
 w = s_1 \circ \cdots \circ s_k, \quad s_i = s_{\alpha_i}, \quad \alpha_i \in \Pi \quad (17)
\]

not necessarily distinct. Then write

\[
 w_i = s_1 \cdots s_i, \quad \text{(in particular} \quad w_k = w). \quad (18)
\]

\[
 w(\alpha_k) = w_{k-1} \cdot s_k(\alpha_k) = w_{k-1}(-\alpha_k) = -w_{k-1}(\alpha_k). \quad (19)
\]

By the assumption, \( w(\Pi) = \Pi \), so \( w(\alpha_k) > 0 \), and therefore \( w_{k-1} \) maps \( \alpha_k \) to a negative root. In such a case we can show that there is \( t < i \) such that

\[
 w_{k-1} = s_1 \cdots s_{t-1} \ s_{t+1} \cdots s_{k-1} \quad (20)
\]
Let $t$ be the smallest so that

$$\gamma = s_{t+1} \cdots s_{k-1}(\alpha_k) > 0.$$  \hfill (17)

Then since $s_t(\gamma)$ is negative, $\gamma = \alpha_t$. So

$$s_t = (s_{t+1} \cdots s_{k-1})s_k(s_{k-1} \cdots s_{t+1}).$$  \hfill (18)

Plug this into the reduced expression for $w$ to find a strictly shorter expression of $w$ in terms of simple reflections.
We assume that the root system $R$ is irreducible. This means that we cannot decompose $R = R_1 \cup R_2$ disjoint union so that each $R_i$ is a root system. Recall that $\alpha < \beta$ means that either $\beta = \alpha$ or $\beta - \alpha$ is a sum of positive roots. This implies that if $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$, then $\alpha_1 \pm \alpha_2$ cannot be a root. Furthermore, $V = V_1 \oplus V_2$ where $V_i$ are the spans of the roots in $R_i$.

A root is called maximal, if for any other root $\alpha$, $\beta - \alpha$ is a sum of positive roots (or is not a root).

**Lemma 9**

*If $R$ is irreducible, there is a unique maximal root $\beta$. Furthermore, $\beta = \sum_{\gamma \in \Pi} m_\gamma \gamma$ with all $m_\gamma > 0$.***
Proof. Let $\beta$ be a maximal root. Then $\beta = \sum_{\alpha \in \Pi} m_{\alpha} \alpha$, $m_{\alpha} \geq 0$.

Observe that $(\gamma, \beta) \geq 0$ for all $\gamma \in \Pi$. Otherwise $(\gamma, \beta) < 0$ implies $\gamma + \beta$ is a root, which is strictly bigger. Define $\Pi_1 := \{\alpha \in \Pi | m_{\alpha} > 0\}$, $\Pi_2 := \{\alpha \in \Pi | m_{\alpha} = 0\}$.

If $\gamma \in \Pi_2$, then $(\gamma, \beta) = \sum_{\alpha \in \Pi_1} m_{\alpha} (\gamma, \alpha) \leq 0$. So $(\gamma, \beta) = 0$ and therefore also $(\gamma, \alpha) = 0$ for all $\alpha \in \Pi_1$. Then $R^+ = R_1^+ \cup R_2^+$, a contradiction. Thus all $m_{\alpha} > 0$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Pi$, and $(\beta, \alpha) > 0$ for at least one simple root.

If $\beta'$ is another maximal root, $(\beta, \beta') > 0$, so $\beta - \beta'$ is a root.

One of $\beta, \beta'$ cannot be maximal.
Lemma 10

The $W$ orbit of any root spans $V$.

Proof.

Follows from previous facts about irreducibility of $V$. See part 2 in proposition 4 in the lecture from April 2-4.

Lemma 11

$R$ has at most 2 root lengths.

Proof.

Let $\alpha_1$, $\alpha_2$ be two roots. We can conjugate one of the roots by the Weyl group so that they have nonzero inner product. The classification of root systems of rank 2 shows that the ratio of the lengths of such roots is either 2 or 3. If for example $\|\alpha_2\|^2 = 2\|\alpha_1\|^2$ and $\|\alpha_3\|^2 = 3\|\alpha_1\|^2$ then $\|\alpha_3\|^2 = \frac{3}{2}\|\alpha_2\|^2$, a contradiction.
Lemma 12

The maximal root is long.

Proof.

Suppose $\beta$ is maximal and $\alpha$ is arbitrary. We want to show $||\alpha||^2 \leq ||\beta||^2$. We can conjugate $\alpha$ by $W$ so that it satisfies $(\alpha, \gamma) \geq 0$ for all $\gamma \in \Pi$. At least one $(\alpha, \gamma) > 0$. Then $(\beta, \alpha) > 0$, and $\beta - \alpha$ must be a root in $R^+$, so a sum of simple roots with nonnegative coefficients. Therefore $(\alpha, \beta - \alpha) \geq 0$, and

$$(\beta, \beta) = (\beta - \alpha, \beta - \alpha) + 2(\alpha, \beta - \alpha) + (\alpha, \alpha) \geq (\alpha, \alpha). \quad (19)$$
The irreducible (or simple) root systems are characterized by their corresponding Cartan matrices.

**Definition 13 (Cartan Matrix)**

An $n \times n$ matrix with integer entries $A = [a_{ij}]$ is called a generalized Cartan matrix if

1. $a_{ii} = 2$,
2. $a_{ij} \leq 0$ for $i \neq j$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

It is called a Cartan matrix if in addition $\det A \neq 0$.

If $R$ is an irreducible root system, then the matrix $\langle \hat{\alpha}_i, \alpha_j \rangle$ is a Cartan matrix.
Proposition 2

*Two root systems with the same Cartan matrix are isomorphic.*

A Cartan matrix gives rise to a **Coxeter-Dynkin Graph**. This has one vertex for each simple root $\alpha_1 \cdots \alpha_\ell$. Two of them are joined if $\langle \alpha_i, \alpha_j \rangle < 0$ as many times as the ratio of lengths. The arrow points to shorter root. The classification of the Dynkin diagrams is described in Humphreys chapter III.