Supplementary problems

Problem 1. Show that \( f(z) = \begin{cases} \frac{e^{iz} - 1}{z} & z \neq 0, \\ i & z = 0 \end{cases} \) is analytic in the plane.

Problem 2. Show that
\[
\int_{0}^{\infty} \frac{\sin t}{t} \, dt := \lim_{R \to \infty} \int_{0}^{R} \frac{\sin t}{t} \, dt
\]
can be written as the limit
\[
\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{it} - 1}{t} \, dt
\]

Problem 3. Evaluate the integral \( \int_{0}^{\infty} \frac{\sin t}{t} \, dt \) by considering the integral of \( \frac{e^{it} - 1}{t} \) over the closed contour consisting of the half-circle \( z = Re^{i\theta} \) for \( 0 \leq \theta \leq \pi \), and its diameter.

Problem 4. Let \( D_r \) be the half-circle \( z = re^{i\theta} \) for \( 0 \leq \theta \leq \pi \), and let \( a_1, \ldots, a_n \) be positive real numbers, and \( A_1, \ldots, A_n \in \mathbb{R} \) such that \( A_1 + \cdots + A_n = 0 \). Calculate the limit of the following integral when \( r \to 0 \), and \( r \to \infty \).
\[
\int_{D_r} \frac{A_1 e^{ia_1 z} + \cdots + A_n e^{ia_n z}}{z^2} \, dz.
\]

Problem 5. With the same notation as in problem 4, evaluate the integral
\[
\int_{0}^{\infty} \frac{A_1 \cos a_1 t + \cdots + A_n \cos a_n t}{t^2} \, dt.
\]