Supplementary problems

Problem 1. Show that $f(z) = \begin{cases} \frac{e^{iz}-1}{z} & z \neq 0, \\ i & z = 0 \end{cases}$ is analytic in the plane.

Problem 2. Show that

$$\int_0^\infty \frac{\sin t}{t}\ dt := \lim_{R\to\infty} \int_0^R \frac{\sin t}{t}\ dt$$

can be written as the limit

$$\lim_{R\to\infty} \int_{-R}^R \frac{e^{it}-1}{t}\ dt$$

Problem 3. Evaluate the integral $\int_0^\infty \frac{\sin t}{t}\ dt$ by considering the integral of $\frac{e^{it}-1}{t}$ over the closed contour consisting of the half-circle $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$, and its diameter.

Problem 4. Let D_r be the half-circle $z = re^{i\theta}$ for $0 \leq \theta \leq \pi$, and let $a_1, \ldots, a_n$ be positive real numbers, and $A_1, \ldots, A_n \in \mathbb{R}$ such that $A_1 + \cdots + A_n = 0$. Calculate the limit of the following integral when $r \to 0$, and $r \to \infty$.

$$\int_{D_r} \frac{A_1 e^{ia_1 z} + \cdots + A_n e^{ia_n z}}{z^2}\ dz.$$

Problem 5. With the same notation as in problem 4, evaluate the integral

$$\int_0^\infty \frac{A_1 \cos a_1 t + \cdots + A_n \cos a_n t}{t^2}\ dt.$$