Problem 1. Let $w = P(z)$ be a polynomial with complex coefficients, and let $z_1, \dots, z_n$ be the roots of the polynomial, with multiplicities. Show that

$$\frac{P'(z)}{P(z)} = \sum \frac{1}{z - z_k}.$$

Problem 2. [Gauss] Suppose ζ is a root of $P'(z) = 0$, with $\zeta \neq z_k$ for any k. Show that ζ is in the convex hull of the z_k.

Remarks/Hints: If $z_1, \dots, z_\ell$ are points in the complex plane, the convex hull is the set

$$\{\sum \lambda_i z_i \ : \ \lambda_i \geq 0, \quad \sum \lambda_i = 1\}.$$

Use the identity

$$\frac{1}{z - z_k} = \frac{\overline{z - z_k}}{|z - z_k|^2}.$$

Poblem 3. [Landau] Show that, for $n \geq 2$, the equation $1 + z + az^n = 0$ always has a solution of absolute value ≤ 2, for any complex number a. (Hint: Replace z by $1/z$.)